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ON FINITE ELASTIC DEFORMATIONS WITH A PERTURBED STRAIN-ENERGY FUNCTION†

By A. J. M. SPENCER‡ (*Brown University*)

[Received 20 February 1958]

SUMMARY

In order to obtain analytical solutions to problems involving finite deformations of elastic solids it is often necessary to adopt a specific form for the strain-energy function. In this paper it is supposed that such a strain-energy function W is perturbed to a new strain-energy function $W+\epsilon W'$, with the result that an additional small deformation is superimposed on the existing finite deformation. Stress-strain relations and equations of equilibrium are formulated which, with suitable boundary conditions and a knowledge of the original finite deformation, make possible the determination of the displacements and stress-components that are superimposed when W is perturbed. The theory is illustrated by application to the problem of simultaneous extension, inflation, and shear of a cylindrical tube, and also to the special case in which the tube undergoes shear only. In these applications W is supposed to have the Mooney form, while W' is an arbitrary function of the strain invariants.

1. Introduction

In recent years a number of solutions have been obtained to problems involving finite deformations of elastic solids. In some particular problems it is possible to obtain complete solutions determining the deformation and state of stress in a body without specifying any particular form for the strain-energy function of the elastic solid. In other cases, the equations of equilibrium are such that they can only be integrated if a specific form of the strain-energy function is assumed. This situation arises, for example, in the simultaneous extension, inflation, and shear of a cylindrical tube. In such cases, the forms most often adopted for the strain-energy function W are the form postulated by Mooney (**1**),

$$W = C_1(I_1-3)+C_2(I_2-3), \tag{1.1}$$

and the neo-Hookean form

$$W = C_1(I_1-3). \tag{1.2}$$

Here I_1 and I_2 are invariants of the strain tensor to be defined later. C_1 and C_2 are positive constants.

Experiments on vulcanized rubbers by Rivlin and Saunders (**2**) and by

† The results presented in this paper were obtained in the course of research sponsored by the Office of Ordnance Research, Department of the Army, under contract DA-19-020-ORD-3487.

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1. Introduction

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Experiments on vulcanized rubbers by Rivlin and Saunders (**2**) and by

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Gent and Rivlin (**3**) show that for these materials the strain-energy function departs slightly from the form (1.1) and for moderately large deformations may be expressed by a formula of the type

$$W = C_1(I_1-3)+f(I_2-3), \tag{1.3}$$

where f is a decreasing function of (I_2-3). It is therefore of interest to investigate the theory of the deformation of elastic materials with a strain-energy function of the form

$$W^* = W+\epsilon W', \tag{1.4}$$

where W is a strain-energy function for which, in a given problem, an explicit solution can be found, and $\epsilon W'$ represents a small perturbing strain-energy function. By studying this form we may to some extent remove the restrictions imposed when the exact solution of the equations requires the assumption of some special form for the strain-energy function. With a suitable choice of W and W', the theory we shall develop may also be applicable to the behaviour of certain real materials. In applications, W will usually have the form (1.1) or (1.2), but the theory does not restrict W to any particular form.

A brief summary of the notation and equations of the theory of finite elastic deformations is presented in section 2. This is followed, in section 3, by a consideration of the effect of replacing the strain-energy function W of a deformed elastic body B by a new strain-energy function $W+\epsilon W'$. This must, in general, result in a further deformation of the body into a second deformed body B'. It is supposed that the deformation and the state of stress of the body B is completely determined, and we assume that the additional displacements that occur when B is deformed to B' are everywhere of order ϵ. The theory then closely resembles the theory of small elastic deformations superposed on finite elastic deformations which has been formulated by Green, Rivlin, and Shield (**4**). Stress-strain relations and equations of equilibrium are derived which, together with suitable boundary conditions and a knowledge of the state of the body B, make possible the determination of the additional displacements and stress components that are superimposed when W is perturbed.

In section 4 the theory is applied to the problem of simultaneous extension, inflation, and shear of a cylindrical tube. The solution of this problem for the case of a Mooney solid is due to Rivlin (**5**). In the present application the additional small displacements and increments to the components of the stress tensor that arise when the material has a strain-energy function of the form (1.4) are calculated in terms of the scalar invariants $\partial W'/\partial I_1$ and $\partial W'/\partial I_2$, and integrals involving these invariants. The increments to the surface tractions that are required to maintain the

deformation are also calculated. In section 5 the special case in which the tube undergoes shear but no extension or inflation is treated in a similar manner.

2. Notation and formulae for finite deformations

The notation adopted is that used by Green and Zerna (**6**), where a full account of the derivation of the theory will be found. For convenience, the principal formulae and notations we shall employ are summarized here.

The points P_0 of an undeformed elastic body B_0, at rest at time $t = 0$, are defined by a system of rectangular Cartesian coordinates x_i, or by a system of general curvilinear coordinates θ_i. The line element ds_0 of B_0 is given by

$$ds_0^2 = dx_i\,dx_i = g_{ik}\,d\theta_i\,d\theta_k. \tag{2.1}$$

g_{ik} is the covariant metric tensor of the unstrained body B_0. The contravariant metric tensor of B_0 is denoted g^{ik} and is defined by

$$g^{ir}g_{rk} = \delta^i_k. \tag{2.2}$$

The usual summation convention is used and indices take the values 1, 2, and 3. The Kronecker delta is denoted by δ^i_k.

The body B_0 is now deformed into a strained body B, so that at time t the material particle which was originally at the point P_0 has moved to the point P. The curvilinear coordinates θ_i move with the body as it is deformed, and form a curvilinear coordinate system in B at time t, in terms of which coordinate system the line element ds in B is given by

$$ds^2 = G_{ik}\,d\theta_i\,d\theta_k. \tag{2.3}$$

G_{ik} is the covariant metric tensor of the strained body B at time t. The contravariant metric tensor of B at time t is G^{ik}, where

$$G^{ir}G_{rk} = \delta^i_k. \tag{2.4}$$

The vector $\overrightarrow{P_0P}$ is the displacement vector, denoted $\mathbf{v}(\theta_1, \theta_2, \theta_3, t)$, and $\mathbf{r}(\theta_1, \theta_2, \theta_3)$ is the position vector of P_0 referred to the origin of the x_i coordinate system. The covariant and contravariant base vectors $\mathbf{G}_i$, $\mathbf{G}^i$ at P of the coordinate system θ_i are then defined by

$$\mathbf{G}_i = \mathbf{r}_{,i} + \mathbf{v}_{,i}, \qquad \mathbf{G}^i = G^{ik}\mathbf{G}_k, \tag{2.5}$$

and we have
$$\mathbf{G}_i.\mathbf{G}_k = G_{ik}, \qquad \mathbf{G}^i.\mathbf{G}^k = G^{ik}. \tag{2.6}$$

A comma denotes partial differentiation with respect to the θ_i coordinates.

The most convenient form of the equations of motion is

$$\tau^{ik}\|_i + \rho F^k = \rho f^k, \tag{2.7}$$

where τ^{ik} is the contravariant stress tensor referred to the coordinates θ_i,

ρ is the density of the strained body B and F^k and f^k are the contravariant components of the body force vector and acceleration vector respectively, both measured per unit mass of B and referred to the base vectors $\mathbf{G}_i$. The double line denotes covariant differentiation with respect to B, that is, with respect to θ^i coordinates and the metric tensor components G^{ik}, G_{ik}. The Christoffel symbols of B are given by

$$\Gamma^r_{ik} = \tfrac{1}{2}G^{rs}(G_{si,k}+G_{sk,i}-G_{ik,s}). \tag{2.8}$$

When the body B_0 is homogeneous and isotropic the strain-energy W, measured per unit volume of the unstrained body B_0, is a function of the strain invariants I_1, I_2, I_3, so that

$$W = W(I_1, I_2, I_3), \tag{2.9}$$

where

$$I_1 = g^{rs}G_{rs}, \quad I_2 = g_{rs}G^{rs}I_3, \quad I_3 = G/g, \quad G = |G_{ik}|, \quad g = |g_{ik}|. \tag{2.10}$$

The stress-strain relations then take the form

$$\tau^{ik} = \Phi g^{ik}+\Psi B^{ik}+pG^{ik}, \tag{2.11}$$

where
$$\Phi = \frac{2}{I_3^{\frac{1}{2}}}\frac{\partial W}{\partial I_1}, \qquad \Psi = \frac{2}{I_3^{\frac{1}{2}}}\frac{\partial W}{\partial I_2}, \qquad p = 2I_3^{\frac{1}{2}}\frac{\partial W}{\partial I_3}, \tag{2.12}$$

$$B^{ik} = (g^{ik}g^{rs}-g^{ir}g^{ks})G_{rs}. \tag{2.13}$$

If the material is also incompressible, then $G = g$, and $I_3 = 1$, at all points of the body, and W is a function of I_1 and I_2 only. The stress-strain relation (2.11) remains valid, but now

$$\Phi = 2\frac{\partial W}{\partial I_1}, \qquad \Psi = 2\frac{\partial W}{\partial I_2}, \tag{2.14}$$

and p is a scalar invariant function of the coordinates θ_i.

When the surface force vector $\mathbf{P}$ is prescribed at a surface with unit outward normal vector $\mathbf{n}$, then at that surface

$$\tau^{ik}n_i = P^k, \tag{2.15}$$

where
$$\mathbf{P} = P^k\mathbf{G}_k, \qquad \mathbf{n} = n_i\mathbf{G}^i.$$

3. General theory

We will now consider the effect on the stresses and displacements in the body B of replacing the strain-energy function W of the previous section by a perturbed strain-energy function $W+\epsilon W'$. It is assumed that, with some given strain-energy function W, a solution of the equations of section 2 has been obtained for some body, which in its initial and first deformed states is denoted by B_0 and B respectively. This is to say that the displacement vector $\mathbf{v}$ and the stress tensor τ^{ik}, which describe

the deformation and the state of stress of the body B, have been determined as functions of a set of curvilinear coordinates θ_i and the derivatives of $W(I_1, I_2, I_3)$, and that these quantities, together with the metric tensors and base vectors of B_0 and B, may be regarded as known in all that follows.

Now suppose that the strain-energy function W is changed to a new strain-energy function $W+\epsilon W'$, where ϵ is a small constant. This will result in a further deformation of the body B into a second deformed body which will be denoted B'. In this further deformation, the points which were at P_0 in B_0, and at P in B, are displaced to P' in B'. The prescribed surface tractions on the surface of the body B' are allowed to differ by quantities of at most order ϵ from the values they have on the surface of the body B. It is assumed that under these conditions the incremental displacement vector $\overrightarrow{PP'}$ will be of order ϵ at every point of the body, so that we may write

$$\overrightarrow{P_0P'} = \overrightarrow{P_0P}+\epsilon\mathbf{w} = \mathbf{v}(\theta_i, t)+\epsilon\mathbf{w}(\theta_i, t), \tag{3.1}$$

and in the following analysis neglect ϵ^2 and higher powers of ϵ in comparison with unity.

As far as the treatment of the geometry of the deformation is concerned, the theory which follows is identical with the theory of small deformations superposed on finite deformations which has been described by Green, Rivlin, and Shield (4). We may therefore make immediate use of certain results of this paper, which are summarized in the following paragraph.

The covariant base vectors of the coordinate system θ_i at points P' of the body B' are denoted by $\mathbf{G}_i+\epsilon\mathbf{G}'_i$, where

$$\mathbf{G}'_i = \mathbf{w}_{,i}. \tag{3.2}$$

The displacement vector $\mathbf{w}$ is most conveniently expressed in components referred to the base vectors $\mathbf{G}_i$, $\mathbf{G}^i$, as follows:

$$\mathbf{w} = w_m\mathbf{G}^m = w^m\mathbf{G}_m, \tag{3.3}$$

so that

$$\mathbf{G}'_i = w_m\|_i\mathbf{G}^m = w^m\|_i\mathbf{G}_m. \tag{3.4}$$

The covariant metric tensor of B', evaluated at time t, is $G_{ik}+\epsilon G'_{ik}$, where

$$G'_{ik} = w_i\|_k+w_k\|_i, \tag{3.5}$$

and the contravariant metric tensor of B' at time t is $G^{ik}+\epsilon G'^{ik}$, where

$$G'^{ik} = -G^{ir}G^{ks}G'_{rs}. \tag{3.6}$$

The determinant of the metric tensor components $G_{ik}+\epsilon G'_{ik}$ is denoted by $G+\epsilon G'$, where

$$G' = GG^{ik}G'_{ik}. \tag{3.7}$$

Finally, the strain invariants associated with the body B' are denoted by $I_1+\epsilon I'_1$, $I_2+\epsilon I'_2$, $I_3+\epsilon I'_3$, where

$$I'_1 = g^{rs}G'_{rs}, \quad I'_2 = g_{rs}(G'^{rs}I_3+G^{rs}I'_3), \quad I'_3 = G'/g = I_3\,G^{rs}G'_{rs}. \tag{3.8}$$

For the body B, the strain-energy function has the form $W(I_1, I_2, I_3)$. For the body B', this becomes

$$W(I_1+\epsilon I'_1,\, I_2+\epsilon I'_2,\, I_3+\epsilon I'_3)+\epsilon W'(I_1+\epsilon I'_1,\, I_2+\epsilon I'_2,\, I_3+\epsilon I'_3).$$

The scalar invariants associated with the body B are Φ, Ψ, and p, and are defined by equation (2.12). The scalar invariants in the body B' will be denoted by $\Phi+\epsilon\Phi'$, $\Psi+\epsilon\Psi''$, $p+\epsilon p'$. These scalar invariants in B' are each derived partly from the strain-energy function W, and partly from the strain-energy function W'. The terms depending on

$$W(I_1+\epsilon I'_1,\, I_2+\epsilon I'_2,\, I_3+\epsilon I'_3)$$

are the same functions of $I_1+\epsilon I'_1$, $I_2+\epsilon I'_2$, and $I_3+\epsilon I'_3$ as Φ, Ψ, and p are of I_1, I_2, and I_3, and so, by Taylor's expansion to order ϵ, these terms are

$$\left.\begin{aligned}
&\Phi+\epsilon\left(I'_1\frac{\partial\Phi}{\partial I_1}+I'_2\frac{\partial\Phi}{\partial I_2}+I'_3\frac{\partial\Phi}{\partial I_3}\right)\\
&\Psi+\epsilon\left(I'_1\frac{\partial\Psi}{\partial I_1}+I'_2\frac{\partial\Psi}{\partial I_2}+I'_3\frac{\partial\Psi}{\partial I_3}\right)\\
&p+\epsilon\left(I'_1\frac{\partial p}{\partial I_1}+I'_2\frac{\partial p}{\partial I_2}+I'_3\frac{\partial p}{\partial I_3}\right)
\end{aligned}\right\}. \tag{3.9}$$

To these must be added the invariants derived from the perturbing strain-energy function $\epsilon W'(I_1+\epsilon I'_1, I_2+\epsilon I'_2, I_3+\epsilon I'_3)$. To the first order in ϵ, these terms are

$$\epsilon\frac{2}{I_3^{\frac{1}{2}}}\frac{\partial W'}{\partial I_1}, \quad \epsilon\frac{2}{I_3^{\frac{1}{2}}}\frac{\partial W'}{\partial I_2}, \quad \epsilon\, 2I_3^{\frac{1}{2}}\frac{\partial W'}{\partial I_3}. \tag{3.10}$$

Combining the terms of order ϵ in (3.9) and (3.10), and making use of (2.12), we have the final expressions for Φ', Ψ'', and p', viz.

$$\left.\begin{aligned}
\Phi' &= AI'_1+FI'_2+EI'_3-\frac{\Phi}{2I_3}I'_3+\frac{2}{I_3^{\frac{1}{2}}}\frac{\partial W'}{\partial I_1}\\
\Psi'' &= FI'_1+BI'_2+DI'_3-\frac{\Psi}{2I_3}I'_3+\frac{2}{I_3^{\frac{1}{2}}}\frac{\partial W'}{\partial I_2}\\
p' &= I_3(EI'_1+DI'_2+CI'_3)+\frac{p}{2I_3}I'_3+2I_3^{\frac{1}{2}}\frac{\partial W'}{\partial I_3}
\end{aligned}\right\}, \tag{3.11}$$

where

$$\left.\begin{aligned} A &= \frac{2}{I_3^{\frac{1}{2}}}\frac{\partial^2 W}{\partial I_1^2}, \quad B = \frac{2}{I_3^{\frac{1}{2}}}\frac{\partial^2 W}{\partial I_2^2}, \quad C = \frac{2}{I_3^{\frac{1}{2}}}\frac{\partial^2 W}{\partial I_3^2} \\ D &= \frac{2}{I_3^{\frac{1}{2}}}\frac{\partial^2 W}{\partial I_2 \partial I_3}, \quad E = \frac{2}{I_3^{\frac{1}{2}}}\frac{\partial^2 W}{\partial I_3 \partial I_1}, \quad F = \frac{2}{I_3^{\frac{1}{2}}}\frac{\partial^2 W}{\partial I_1 \partial I_2} \end{aligned}\right\}. \tag{3.12}$$

For an incompressible body, $I_3 = 1$ and $I'_3 = 0$. W and W' are functions of I_1 and I_2 only, so that

$$\Phi = 2\frac{\partial W}{\partial I_1}, \qquad \Psi = 2\frac{\partial W}{\partial I_2}, \tag{3.13}$$

and p is a scalar function of the coordinates which is determined by the equations of motion and boundary conditions for the body B. Also A, F, and B may still be found from (3.12) by setting I_3 equal to unity, and we have

$$\left.\begin{aligned} \Phi' &= AI'_1 + FI'_2 + 2\frac{\partial W'}{\partial I_1} \\ \Psi' &= FI'_1 + BI'_2 + 2\frac{\partial W'}{\partial I_2} \end{aligned}\right\}. \tag{3.14}$$

The pressure function p' cannot now be derived from the elastic potential and is a scalar invariant function which must be determined by the equations of motion and boundary conditions for the body B'.

The remainder of the theory is again formally identical to the theory of small deformations superposed on finite deformations, but it should be noted that the functions Φ', Ψ', and p' of the present theory differ from the functions Φ', Ψ', and p' which appear in the paper by Green, Rivlin, and Shield by the addition of terms involving the perturbing strain-energy function W'.

The tensor B^{ik} becomes, in the body B', a tensor $B^{ik} + \epsilon B'^{ik}$, where

$$B'^{ik} = (g^{ik}g^{rs} - g^{ir}g^{ks})G'_{rs}. \tag{3.15}$$

To the first order in ϵ the stress tensor for the body B', referred to curvilinear coordinates θ_i in B, is $\tau^{ik} + \epsilon\tau'^{ik}$. By substitution in the stress-strain relations, we find

$$\tau'^{ik} = g^{ik}\Phi' + B^{ik}\Psi' + B'^{ik}\Psi + G^{ik}p' + G'^{ik}p. \tag{3.16}$$

If the contravariant components of the body force and acceleration vectors for B' are respectively $F^k + \epsilon F'^k$ and $f^k + \epsilon f'^k$, referred to base vectors $\mathbf{G}_k + \epsilon\mathbf{G}'_k$, and ρ is the density of B, then the equations of motion for B' are

$$[\tau'^{ik} + \tau^{ik}w^r\|_r + \tau^{ir}w^k\|_r]\,\|_i + \rho[F'^k + F^r w^k\|_r] = \rho[f'^k + f^r w^k\|_r]. \tag{3.17}$$

In the case in which surface forces are prescribed on B', and $n_i+\epsilon n'_i$ are the covariant components of the unit normal to the surface of B', referred to base vectors $\mathbf{G}^i+\epsilon\mathbf{G}'^i$, while $P^k+\epsilon P'^k$ are the components of the surface force vector, then the boundary condition at the surface of B' is

$$(\tau^{ik}+\epsilon\tau'^{ik})(n_i+\epsilon n'_i) = P^k+\epsilon P'^k. \tag{3.18}$$

4. Shear, extension, and inflation of a cylindrical tube

As an example of the application of the preceding theory, we now consider the simultaneous shear, extension, and inflation of an incompressible cylindrical tube. This problem has been solved in the case in which the strain-energy function of the body has the Mooney form by Rivlin (5), and by a rather different method by Green and Zerna (6). We use here the method and notation of Green and Zerna.

A circular cylindrical tube which in its unstressed state has external and internal radii a_1 and a_2, and length l, is subjected to the following successive deformations:

(i) A uniform simple extension of extension ratio λ.

(ii) A uniform inflation of the tube, in which the outer and inner radii of the tube change to $r_1 = \mu_1 a_1$ and $r_2 = \mu_2 a_2$, respectively, the length remaining unaltered.

(iii) A shear of the tube about its axis, in which each element of the tube rotates about the axis through an angle ϕ which depends only on the radial position of the element.

(iv) A shear in the direction parallel to the axis, in which each element moves in a direction parallel to the axis through a distance w which depends only on the radial position of the element.

The moving coordinates $(\theta_1, \theta_2, \theta_3)$ are chosen so that they coincide with polar coordinates in the deformed body. The point (r, θ, z) in the deformed body was originally at the point (ϖ, ψ, ζ), where

$$\varpi = \varpi(r) = rQ(r), \quad \theta = \psi+\phi(r), \quad z = \lambda\zeta+w(r), \tag{4.1}$$

and hence, in the notation defined in section 2,

$$x_1 = rQ(r)\cos(\theta-\phi), \quad x_2 = rQ(r)\sin(\theta-\phi), \quad x_3 = [z-w(r)]/\lambda. \tag{4.2}$$

The theory outlined in section 2 may be applied without making any assumption regarding the form of the strain-energy function, as far as the formulation of the equations of equilibrium of the deformed cylinder. This is done by Rivlin and by Green and Zerna. The following results are quoted from Green and Zerna. The metric tensors of the undeformed

body are

$$g_{ik} = \begin{bmatrix} \dfrac{\lambda^2}{Q^2}+r^2Q^2\phi_r^2+\dfrac{w_r^2}{\lambda^2} & -r^2Q^2\phi_r & -\dfrac{w_r}{\lambda^2} \\ -r^2Q^2\phi_r & r^2Q^2 & 0 \\ -\dfrac{w_r}{\lambda^2} & 0 & \dfrac{1}{\lambda^2} \end{bmatrix}, \tag{4.3}$$

$$g^{ik} = \begin{bmatrix} \dfrac{Q^2}{\lambda^2} & \dfrac{Q^2\phi_r}{\lambda^2} & \dfrac{Q^2w_r}{\lambda^2} \\ \dfrac{Q^2\phi_r}{\lambda^2} & \dfrac{1}{r^2Q^2}+\dfrac{Q^2\phi_r^2}{\lambda^2} & \dfrac{Q^2\phi_r w_r}{\lambda^2} \\ \dfrac{Q^2w_r}{\lambda^2} & \dfrac{Q^2\phi_r w_r}{\lambda^2} & \lambda^2+\dfrac{Q^2w_r^2}{\lambda^2} \end{bmatrix}, \tag{4.4}$$

where the suffix r denotes differentiation with respect to the r coordinate. The metric tensors of the deformed body are

$$G_{ik} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad G^{ik} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{r^2} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.5}$$

In equations (4.3) and (4.4) the expressions for the components of g_{ik} and g^{ik} have been simplified by use of the incompressibility condition

$$g = G = r^2,$$

which determines Q as a function of r by the equation

$$rQ = \{\lambda(r^2+K)\}^{\frac{1}{2}}, \tag{4.6}$$

where
$$\lambda K = a_1^2(1-\lambda\mu_1^2) = a_2^2(1-\lambda\mu_2^2). \tag{4.7}$$

The strain invariants are

$$\left.\begin{aligned} I_1 &= \frac{1}{Q^2}+\lambda^2+\frac{Q^2}{\lambda^2}+\frac{r^2Q^2\phi_r^2}{\lambda^2}+\frac{Q^2w_r^2}{\lambda^2} \\ I_2 &= Q^2+\frac{1}{\lambda^2}+\frac{\lambda^2}{Q^2}+r^2Q^2\phi_r^2+\frac{w_r^2}{\lambda^2} \end{aligned}\right\}, \tag{4.8}$$

and the tensor B^{ik} is

$$B^{ik} = \begin{bmatrix} \dfrac{1}{\lambda^2}+Q^2 & Q^2\phi_r & \dfrac{w_r}{\lambda^2} \\ Q^2\phi_r & \dfrac{\lambda^2}{r^2Q^2}+\dfrac{1}{\lambda^2r^2}+Q^2\phi_r^2+\dfrac{w_r^2}{\lambda^2r^2} & 0 \\ \dfrac{w_r}{\lambda^2} & 0 & Q^2+\dfrac{\lambda^2}{Q^2}+r^2Q^2\phi_r^2+\dfrac{w_r^2}{\lambda^2} \end{bmatrix}. \tag{4.9}$$

The components of the stress tensor are then shown to be

$$\left.\begin{aligned}
\tau^{11} &= \frac{Q^2}{\lambda^2}\Phi+\left(\frac{1}{\lambda^2}+Q^2\right)\Psi+p \\
r^2\tau^{22} &= \left(\frac{1}{Q^2}+\frac{r^2Q^2\phi_r^2}{\lambda^2}\right)\Phi+\left(\frac{\lambda^2}{Q^2}+\frac{1}{\lambda^2}+r^2Q^2\phi_r^2+\frac{w_r^2}{\lambda^2}\right)\Psi+p \\
\tau^{33} &= \left(\lambda^2+\frac{Q^2w_r^2}{\lambda^2}\right)\Phi+\left(Q^2+\frac{\lambda^2}{Q^2}+r^2Q^2\phi_r^2+\frac{w_r^2}{\lambda^2}\right)\Psi+p \\
\tau^{12} &= \frac{Q^2\phi_r}{\lambda^2}\Phi+Q^2\phi_r\Psi \\
\tau^{31} &= \frac{Q^2w_r}{\lambda^2}\Phi+\frac{w_r}{\lambda^2}\Psi \\
\tau^{23} &= \frac{Q^2\phi_r w_r}{\lambda^2}\Phi
\end{aligned}\right\}. \tag{4.10}$$

The non-zero Christoffel symbols are

$$\Gamma^1_{22} = -r, \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}. \tag{4.11}$$

Assuming zero body forces, and noting that the components of the stress tensor are functions of r only, the equations of equilibrium are found to be

$$\left.\begin{aligned}
\frac{d\tau^{11}}{dr}+\frac{1}{r}(\tau^{11}-r^2\tau^{22}) &= 0 \\
\frac{d\tau^{12}}{dr}+\frac{3}{r}\tau^{12} &= 0 \\
\frac{d\tau^{13}}{dr}+\frac{1}{r}\tau^{13} &= 0
\end{aligned}\right\}. \tag{4.12}$$

The last two of these may be integrated immediately, and on substituting for τ^{12} and τ^{13} they yield

$$\left.\begin{aligned}
\phi_r &= \frac{2B_1}{r^3Q^2(\Phi+\lambda^2\Psi)} \\
w_r &= \frac{2D_1}{r(Q^2\Phi+\Psi)}
\end{aligned}\right\}, \tag{4.13}$$

where B_1 and D_1 are constants of integration.

In general, Φ and Ψ are functions of I_1 and I_2, which are shown by equation (4.8) to be dependent on ϕ, w, and r, so that equations (4.13) are non-linear differential equations which determine ϕ and w as functions of r. To integrate these equations we must assume some definite form for the strain-energy function W. If we suppose W has the Mooney

form, defined by equation (1.1), then

$$\Phi = 2C_1, \qquad \Psi = 2C_2,$$

and the equations of equilibrium can be integrated to give

$$\left.\begin{aligned} \phi &= -\frac{B_1}{\lambda K(C_1+\lambda^2 C_2)}\log\frac{Q}{Q_2}+B_2 \\ w &= \frac{D_1}{2(\lambda C_1+C_2)}\log\left\{\frac{(C_1 Q^2+C_2)r^2}{(C_1 Q_2^2+C_2)r_2^2}\right\}+D_2 \\ p &= (2\lambda\log Q-Q^2)\left\{\frac{C_1}{\lambda^2}+C_2+\frac{B_1^2}{\lambda^4K^2(C_1+\lambda^2C_2)}\right\}- \\ &\qquad -\frac{D_1^2C_2}{\lambda^2r^2(C_1Q^2+C_2)(\lambda C_1+C_2)}-\frac{2C_2}{\lambda^2}+F \end{aligned}\right\}, \tag{4.14}$$

where $Q_2 r_2 = a_2$, F is an arbitrary constant hydrostatic pressure, and B_1, B_2, D_1, and D_2 are constants of integration which are determined by the conditions at the curved surfaces of the tube. For example, if $w = \phi = 0$ at $r = r_2$, and $w = w_0$, $\phi = \phi_0$ at $r = r_1$, we find

$$B_1 = -\frac{\lambda K(C_1+\lambda^2C_2)\phi_0}{\log(Q_1/Q_2)}, \qquad D_1 = 2(\lambda C_1+C_2)w_0\Big/\log\left\{\frac{(C_1Q_1^2+C_2)r_1^2}{(C_1Q_2^2+C_2)r_2^2}\right\},$$

$$B_2 = D_2 = 0, \tag{4.15}$$

where $Q_1 r_1 = a_1$.

The surface tractions may now readily be determined by the application of equation (2.15). The expressions for the surface tractions are given by Rivlin and by Green and Zerna.

Now consider the modifications that must be made if the strain-energy function W is changed to $W+\epsilon W'$. An additional deformation of the body will then in general be superimposed on the previous deformation. As the radial symmetry of the body is unaltered, this additional deformation will depend only on the coordinate r. We suppose that in addition to the previous finite deformations (i), (ii), (iii), and (iv), the cylinder now undergoes the following small deformations:

(v) A shear of the tube about its axis, in which each element rotates through an angle $\epsilon\phi'$, which is a function of r only.

(vi) A shear parallel to the axis, in which each element moves through a distance $\epsilon w'$, which is a function only of the radial position of the element.

Since $Q(r)$ is shown by (4.6) and (4.7) to be independent of the form of the strain-energy function, perturbing the strain-energy function causes

no additional displacements in the radial direction provided that the internal and external radii of the tube are kept constant. The displacement vector **w** therefore has components in the θ direction, determined by (v), and in the z-direction, determined by (vi), but none in the radial direction.

With the above assumptions about the form of the deformation, the displacement vector **w** has, in the notation of section 3, the following components referred to the base vectors of the polar coordinates (r, θ, z) in the first deformed body:

$$\left.\begin{aligned} w^1 &= 0, \qquad w^2 = \phi'(r), \qquad w^3 = w'(r) \\ w_1 &= 0, \qquad w_2 = r^2\phi'(r), \qquad w_3 = w'(r) \end{aligned}\right\}. \tag{4.16}$$

Hence, from (3.5) and (3.6),

$$G'_{ik} = \begin{bmatrix} 0 & r^2\phi'_r & w'_r \\ r^2\phi'_r & 0 & 0 \\ w'_r & 0 & 0 \end{bmatrix}, \qquad G'^{ik} = \begin{bmatrix} 0 & -\phi'_r & -w'_r \\ -\phi'_r & 0 & 0 \\ -w'_r & 0 & 0 \end{bmatrix}, \tag{4.17}$$

and from (3.8)

$$\left.\begin{aligned} I'_1 &= 2\left\{\frac{Q^2r^2}{\lambda^2}\phi_r\phi'_r + \frac{Q^2}{\lambda^2}w_r w'_r\right\} \\ I'_2 &= 2\left\{Q^2r^2\phi_r\phi'_r + \frac{1}{\lambda^2}w_r w'_r\right\} \\ I'_3 &= 0 \end{aligned}\right\}, \tag{4.18}$$

where the suffix r again denotes differentiation with respect to r. It follows, from (3.14) that

$$\left.\begin{aligned} \Phi' &= 2A\left[\frac{Q^2r^2}{\lambda^2}\phi_r\phi'_r + \frac{Q^2}{\lambda^2}w_r w'_r\right] + 2F\left[Q^2r^2\phi_r\phi'_r + \frac{1}{\lambda^2}w_r w'_r\right] + 2\frac{\partial W'}{\partial I_1} \\ \Psi' &= 2F\left[\frac{Q^2r^2}{\lambda^2}\phi_r\phi'_r + \frac{Q^2}{\lambda^2}w_r w'_r\right] + 2B\left[Q^2r^2\phi_r\phi'_r + \frac{1}{\lambda^2}w_r w'_r\right] + 2\frac{\partial W'}{\partial I_2} \end{aligned}\right\}. \tag{4.19}$$

Substituting in (3.15), the tensor B'^{ik} is seen to be

$$B'^{ik} = \begin{bmatrix} 0 & -\dfrac{\phi'_r}{\lambda^2} & -Q^2w'_r \\ -\dfrac{\phi'_r}{\lambda^2} & 2\dfrac{w_r w'_r}{r^2\lambda^2} & -\dfrac{1}{\lambda^2}w_r\phi'_r - Q^2\phi_r w'_r \\ -Q^2w'_r & -\dfrac{1}{\lambda^2}w_r\phi'_r - Q^2\phi_r w'_r & 2Q^2r^2\phi_r\phi'_r \end{bmatrix}, \tag{4.20}$$

and, applying (3.16), the incremental stress tensor τ'^{ik} has components

$$\left.\begin{aligned}
\tau'^{11} &= \frac{Q^2}{\lambda^2}\Phi' + \left(\frac{1}{\lambda^2}+Q^2\right)\Psi'' + p' \\
r^2\tau'^{22} &= \left(\frac{1}{Q^2}+\frac{r^2Q^2\phi_r^2}{\lambda^2}\right)\Phi' + \left(\frac{\lambda^2}{Q^2}+\frac{1}{\lambda^2}+r^2Q^2\phi_r^2+\frac{w_r^2}{\lambda^2}\right)\Psi'' + \\
&\qquad + \frac{2w_r w_r'}{\lambda^2}\Psi + p' \\
\tau'^{33} &= \left(\lambda^2+\frac{Q^2w_r^2}{\lambda^2}\right)\Phi' + \left(Q^2+\frac{\lambda^2}{Q^2}+r^2Q^2\phi_r^2+\frac{w_r^2}{\lambda^2}\right)\Psi'' + \\
&\qquad + 2Q^2r^2\phi_r\phi_r'\Psi + p' \\
\tau'^{23} &= \frac{Q^2\phi_r w_r}{\lambda^2}\Phi' - \left(\frac{1}{\lambda^2}w_r\phi_r' + Q^2\phi_r w_r'\right)\Psi \\
\tau'^{31} &= \frac{Q^2 w_r}{\lambda^2}\Phi' + \frac{w_r}{\lambda^2}\Psi'' - Q^2 w_r'\Psi - w_r' p \\
\tau'^{12} &= \frac{Q^2\phi_r}{\lambda^2}\Phi' + Q^2\phi_r\Psi'' - \frac{\phi_r'}{\lambda^2}\Psi - \phi_r' p
\end{aligned}\right\}. \tag{4.21}$$

Since by symmetry the stress components are functions of r alone, the equations of equilibrium (3.17) become

$$\left.\begin{aligned}
&\frac{d}{dr}[\tau'^{11}-r\tau^{12}\phi'] + \frac{1}{r}[\tau'^{11}-r^2\tau'^{22}-\tau^{12}(2r\phi'+r^2\phi_r')] = 0 \\
&\frac{d}{dr}\left[\tau'^{12}+\tau^{11}\left(\phi_r'+\frac{1}{r}\phi'\right)\right] + \frac{1}{r}\left[3\tau'^{12}+2\tau^{11}\left(\phi_r'+\frac{1}{r}\phi'\right)-r\tau^{22}\phi'\right] = 0 \\
&\frac{d}{dr}[\tau'^{31}+\tau^{11}w_r'] + \frac{1}{r}[\tau'^{31}+\tau^{11}w_r'] = 0
\end{aligned}\right\}. \tag{4.22}$$

Making use of the first two of equations (4.12), the equations of equilibrium (4.22) may be rewritten

$$\left.\begin{aligned}
&\frac{d\tau'^{11}}{dr} + \frac{1}{r}[\tau'^{11}-r^2\tau'^{22}] - 2r\tau^{12}\phi_r' = 0 \\
&\frac{d}{dr}[\tau'^{12}+\tau^{11}\phi_r'] + \frac{3}{r}[\tau'^{12}+\tau^{11}\phi_r'] = 0 \\
&\frac{d}{dr}[\tau'^{31}+\tau^{11}w_r'] + \frac{1}{r}[\tau'^{31}+\tau^{11}w_r'] = 0
\end{aligned}\right\}. \tag{4.23}$$

The last two of these may be integrated immediately, giving

$$\left.\begin{aligned}
\tau'^{12}+\tau^{11}\phi_r' &= \frac{2B_1'}{r^3} \\
\tau'^{31}+\tau^{11}w_r' &= \frac{2D_1'}{r}
\end{aligned}\right\}, \tag{4.24}$$

where B_1' and D_1' are constants of integration.

At this stage it is necessary to assume a particular form for the strain-energy function W. We may, however, proceed without making any assumption about $\epsilon W'$, other than that ϵ is small. If we suppose that W has the Mooney form, then

$$\Phi = 2C_1, \qquad \Psi = 2C_2, \qquad A = B = F = 0,$$

$$\Phi' = 2\frac{\partial W'}{\partial I_1}, \qquad \Psi' = 2\frac{\partial W'}{\partial I_2},$$

and ϕ, w, and p are given by equations (4.14). Equations (4.24) then become, after substituting for the stress components,

$$\left.\begin{aligned} \phi'_r &= \frac{1}{(C_1+\lambda^2 C_2)Q^2 r^3}\left\{\lambda^2 B'_1 - \tfrac{1}{2}B_1\frac{\Phi'+\lambda^2\Psi'}{C_1+\lambda^2 C_2}\right\} \\ w'_r &= \frac{1}{r(Q^2C_1+C_2)}\left\{\lambda^2 D'_1 - \tfrac{1}{2}D_1\frac{Q^2\Phi'+\Psi'}{Q^2C_1+C_2}\right\} \end{aligned}\right\}, \tag{4.25}$$

so that

$$\left.\begin{aligned} \phi' &= \frac{-\lambda B'_1}{K(C+\lambda^2 C_2)}\log\frac{Q}{Q_2} + \frac{B_1}{2\lambda K(C_1+\lambda^2 C_2)^2}\int\limits_{Q_2}^{Q}(\Phi'+\lambda^2\Psi')\frac{dQ}{Q} + B'_2 \\ w' &= \frac{\lambda^2 D'_1}{4(C_1+\lambda^2 C_2)}\log\left\{\frac{(C_1 Q^2+C_2)r^2}{(C_1 Q_2^2+C_2)r_2^2}\right\} - \tfrac{1}{2}D_1\int\limits_{r_2}^{r}\frac{(Q^2\Phi'+\Psi')\,dr}{(Q^2C_1+C_2)^2 r} + D'_2 \end{aligned}\right\}, \tag{4.26}$$

where B'_2 and D'_2 are constants of integration.

If we require as boundary conditions that the displacements at the curved surface of the second deformed body are to be the same as the displacements at the curved surface of the first deformed body, then

$$\phi' = w' = 0 \quad \text{when } r = r_1 \text{ and } r = r_2,$$

and it follows that $\qquad B'_2 = D'_2 = 0,$

$$\left.\begin{aligned} B'_1 &= \frac{B_1}{2\lambda^2(C_1+\lambda^2 C_2)\log(Q_1/Q_2)}\int\limits_{Q_2}^{Q_1}(\Phi'+\lambda^2\Psi')\frac{dQ}{Q} \\ D'_1 &= \left[2D_1(C_1+\lambda^2 C_2)\Big/\lambda^2\log\left\{\frac{(C_1 Q_1^2+C_2)r_1^2}{(C_1 Q_2^2+C_2)r_2^2}\right\}\right]\int\limits_{r_2}^{r_1}\frac{(Q^2\Phi'+\Psi')\,dr}{(Q^2C_1+C_2)^2 r} \end{aligned}\right\}. \tag{4.27}$$

The first of equations (4.23) then gives, after integration and some

simplification,

$$p' = -\frac{Q^2}{\lambda^2}[\Phi'+\lambda^2\Psi''] - \frac{1}{\lambda^2}\Psi' + \frac{2B_1 B_1'(2\lambda\log Q - Q^2)}{\lambda^2 K^2(C_1+\lambda^2 C_2)} +$$
$$+\frac{2D_1 D_1' C_2}{\lambda K C_1(Q^2C_1+C_2)} - \int^r \left[\left(\frac{Q^2}{\lambda^2}-\frac{1}{Q^2}\right)(\Phi'+\lambda^2\Psi'')+\right.$$
$$\left.+\frac{B_1^2(\Phi'+\lambda^2\Psi'')}{r^4Q^2\lambda^2(C_1+\lambda^2C_2)^2}+\frac{D_1^2\{2C_2Q^2\Phi'-(Q^2C_1-C_2)\Psi''\}}{\lambda^2 r^2(Q^2C_1+C_2)^3}\right]\frac{dr}{r}+F', \tag{4.28}$$

where F' is a constant which represents an arbitrary hydrostatic pressure.

If the equation of the surface of the body B' is

$$F(\theta_1, \theta_2, \theta_3) = 0, \tag{4.29}$$

then the covariant components of the unit outward normal vector $\mathbf{n}$ to B', referred to base vectors $\mathbf{G}^i+\epsilon\mathbf{G}'^i$, are

$$n_i+\epsilon n_i' = k\frac{\partial F}{\partial \theta_i}, \tag{4.30}$$

where k is chosen so that $\mathbf{n}$ is a unit vector. In the present example, the curved surfaces $r = r_1$, $r = r_2$ in B become, in B', with the boundary conditions that have been adopted, the surfaces $\theta_1 = r_1$ and $\theta_1 = r_2$, so that $\mathbf{n}$ has components $(1, 0, 0)$ at the outer curved surface of the cylinder, and $(-1, 0, 0)$ at the inner curved surface. Equation (3.18) then gives as the components of the surface force vectors

$$\left.\begin{aligned} P^k+\epsilon P'^k &= (\tau^{1k})_{r=r_1}+\epsilon(\tau'^{1k})_{r=r_1} \quad \text{at } \theta_1 = r = r_1 \\ P^k+\epsilon P'^k &= -(\tau^{1k})_{r=r_2}-\epsilon(\tau'^{1k})_{r=r_2} \quad \text{at } \theta_1 = r = r_2 \end{aligned}\right\}, \tag{4.31}$$

these components being referred to the base vectors $\mathbf{G}_i+\epsilon\mathbf{G}_i'$.

The ends of the cylinder, which in the body B_0 are plane, are deformed in the body B to the surfaces

$$z\pm\lambda l - w(r) = 0$$

and in the body B' to the surfaces

$$z\pm\lambda l - w(r) - \epsilon w'(r) = 0.$$

Since the coordinates θ_i in the body B' are related to the coordinates r, θ, z by the equations

$$\left.\begin{aligned} r &= \theta_1 \\ \theta &= \theta_2+\epsilon\phi' \\ z &= \theta_3+\epsilon w' \end{aligned}\right\}, \tag{4.32}$$

the equations of the end surfaces of the cylinder may be written in the form

$$\theta_3\pm\lambda l - w(\theta_1) = 0. \tag{4.33}$$

On the surface which corresponds to the positive sign in equation (4.33), the vector **n** has components

$$\left.\begin{aligned}(1+w_r^2)^{\frac{1}{2}}(n_1+\epsilon n_1') &= -w_r\\(1+w_r^2)^{\frac{1}{2}}(n_2+\epsilon n_2') &= 0\\(1+w_r^2)^{\frac{1}{2}}(n_3+\epsilon n_3') &= 1\end{aligned}\right\}, \tag{4.34}$$

and so at this surface

$$(1+w_r^2)^{\frac{1}{2}}(P^k+\epsilon P'^k) = -w_r(\tau^{1k}+\epsilon\tau'^{1k})+(\tau^{3k}+\epsilon\tau'^{3k}), \tag{4.35}$$

where again the components of the surface force vector are referred to base vectors $\mathbf{G}_i+\epsilon\mathbf{G}_i'$.

Alternatively we may express the surface force vectors in components $Q^k+\epsilon Q'^k$ referred to base vectors $\mathbf{G}_i$. In order to do this we transform from the coordinates $(\theta_1, \theta_2, \theta_3)$ to the coordinates (r, θ, z), using the relations (4.32). On making this transformation, we obtain

$$\left.\begin{aligned}Q^1+\epsilon Q'^1 &= P^1+\epsilon P'^1\\Q^2+\epsilon Q'^2 &= P^2+\epsilon(P'^2+\phi_r' P^1)\\Q^3+\epsilon Q'^3 &= P^3+\epsilon(P'^3+w_r' P^1)\end{aligned}\right\}. \tag{4.36}$$

5. Shear of a cylindrical tube

We now consider the special case of the problem of the preceding section in which the tube is not extended or inflated, so that

$$\mu_1 = \mu_2 = \lambda = 1.$$

It follows that $Q = Q_1 = Q_2 = 1$, and $K = 0$. In this case, again following Green and Zerna, the solution of equations (4.13) applied to a Mooney solid, with boundary conditions $\phi = w = 0$ at $r = a_2$, and $\phi = \phi_0$, $w = w_0$ at $r = a_1$; is

$$\phi = -\frac{B_1}{2(C_1+C_2)}\left(\frac{1}{r^2}-\frac{1}{a_2^2}\right), \qquad w = \frac{D_1\log(r/a_2)}{C_1+C_2}, \tag{5.1}$$

and the pressure function p is found to be

$$p = -\frac{B_1^2}{2(C_1+C_2)r^4}-\frac{C_2 D_1^2}{(C_1+C_2)^2 r^2}+F, \tag{5.2}$$

where $$B_1 = \frac{2(C_1+C_2)a_1^2 a_2^2\phi_0}{a_1^2-a_2^2}, \qquad D_1 = \frac{(C_1+C_2)w_0}{\log(a_1/a_2)}. \tag{5.3}$$

The incremental stress components τ'^{ij} are again given by (4.21), with appropriate substitutions for the special case now being considered, and

as before equations (4.23) are the equilibrium equations for the second deformed body. Equations (4.25) now simplify to

$$\left.\begin{aligned} \phi'_r &= \frac{1}{(C_1+C_2)r^3}\left[B'_1-\tfrac{1}{2}B_1\frac{\Phi'+\Psi''}{C_1+C_2}\right] \\ w'_r &= \frac{1}{(C_1+C_2)r}\left[D'_1-\tfrac{1}{2}D_1\frac{\Phi'+\Psi''}{C_1+C_2}\right] \end{aligned}\right\}, \tag{5.4}$$

which on integration give

$$\left.\begin{aligned} \phi' &= \frac{1}{(C_1+C_2)}\left[\tfrac{1}{2}B'_1\left(\frac{1}{a_2^2}-\frac{1}{r^2}\right)-\frac{B_1}{2(C_1+C_2)}\int\limits_{a_2}^{r}(\Phi'+\Psi'')\frac{dr}{r^3}+B'_2\right] \\ w' &= \frac{1}{(C_1+C_2)}\left[D'_1\log\left(\frac{r}{a_2}\right)-\frac{D_1}{2(C_1+C_2)}\int\limits_{a_2}^{r}(\Phi'+\Psi'')\frac{dr}{r}+D'_2\right] \end{aligned}\right\}. \tag{5.5}$$

If we again impose the conditions $w' = \phi' = 0$ when $r = a_1$, $r = a_2$, then

$$\left.\begin{aligned} B'_2 &= D'_2 = 0 \\ B'_1 &= \frac{a_1^2 a_2^2 B_1}{(a_1^2-a_2^2)(C_1+C_2)}\int\limits_{a_2}^{a_1}(\Phi'+\Psi'')\frac{dr}{r^3} \\ D'_1 &= \frac{D_1}{2\log(a_1/a_2)(C_1+C_2)}\int\limits_{a_2}^{a_1}(\Phi'+\Psi'')\frac{dr}{r} \end{aligned}\right\}. \tag{5.6}$$

p' is then found from the first of equations (4.23) to have the value

$$p' = F'-\Phi'-2\Psi''-\frac{B_1 B'_1}{r^4(C_1+C_2)}-\frac{2D_1 D'_1 C_2}{r^2(C_1+C_2)^2}-$$
$$-\int\limits_{a_2}^{r}\left\{\frac{B_1^2}{(C_1+C_2)^2}\frac{(\Phi'+\Psi'')}{r^4}+\frac{D_1^2}{(C_1+C_2)^3}\frac{[2C_2\Phi'-(C_1-C_2)\Psi'']}{r^2}\right\}\frac{dr}{r}. \tag{5.7}$$

Equations (4.31) and (4.35) again give the components of the surface force vector.

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EXACT SOLUTIONS FOR THE GROWTH OF FINGERS FROM A FLAT INTERFACE BETWEEN TWO FLUIDS IN A POROUS MEDIUM OR HELE-SHAW CELL

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SUMMARY

A family of exact solutions of the equations of motion for the two-dimensional unsteady motion of a viscous liquid driven by a fluid of negligible viscosity in a porous medium or Hele-Shaw cell is obtained. The solutions represent the growth of a particular disturbance of a state of uniform motion and its development into a motion in which long fingers of arbitrary width and spacing penetrate the viscous liquid.

1. Introduction

In a recent paper, Saffman and Taylor (**1**) have considered the motion of the interface between two viscous fluids in a porous medium or Hele-Shaw cell when one of the fluids is driven by the other. The flow in a Hele-Shaw cell consisting of two parallel, closely spaced, flat plates is mathematically analogous to two-dimensional flow in a porous medium in which the motion is governed by Darcy's law.

The stability of a steady state of uniform motion, in which the interface is plane and perpendicular to the direction of flow, was investigated both theoretically and experimentally and it was shown that small disturbances of the interface grow if the less viscous fluid drives the more viscous with a velocity greater than a certain value. The later stages of this instability were investigated using a Hele-Shaw cell and were found to consist of the penetration of 'fingers' of the less viscous fluid into the more viscous one. A natural idealization of this phenomenon is to consider the propagation of equal and equally spaced fingers, and since the streamline midway between two such fingers is straight and may be replaced by a channel wall (at least for the purposes of mathematical analysis), this led to a study of the steady propagation of a single long finger of fluid through a parallel-sided channel containing more viscous fluid. The latter motion was studied in some detail in (**1**).

During the course of this work, exact unsteady solutions of the equations of motion were discovered for the case in which interfacial stress effects are negligible and the viscosity of the driving fluid can be neglected. The solutions represent the growth of long fingers propagating steadily through the viscous liquid from a flat interface which is given a particular initial

disturbance. The form of the initial disturbance is approximately sinusoidal when its amplitude is small. The motion is periodic with arbitrary wavelength in a direction perpendicular to that of the velocity at infinity, and the flow in one complete wavelength represents the growth of a single finger in a parallel-sided channel. The solution does not appear in itself to be of much physical importance, since the particular initial conditions cannot be exactly reproduced in practice. However, it is of some mathematical interest, being an exact solution of a non-linear problem, and does have some bearing on a problem of physical interest.

2. The equations of motion

We consider here the motion of a viscous liquid which is driven by a fluid of negligible viscosity in a region of infinite extent. The theory is restricted to the case in which the two fluids do not interpenetrate to any marked extent and can be regarded as separated by a sharp interface; and the analysis applies to two-dimensional motion in a porous medium or flow in a Hele-Shaw cell. For definiteness we consider the latter. Let x and y be coordinates in the plane of the plates bounding the cell, with the x-axis taken parallel to the velocity at infinity; then neglecting the effects of gravity (these can be taken into account without difficulty but they complicate the algebra and it is found that they do not alter the character of the solutions in any way), the mean velocity across the stratum of viscous fluid contained between the plates is given by

$$\begin{aligned} u &= -\frac{b^2}{12\mu}\frac{\partial p}{\partial x} = \frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y}, \\ v &= -\frac{b^2}{12\mu}\frac{\partial p}{\partial y} = \frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x}, \end{aligned} \tag{1}$$

where u and v denote the components of mean velocity parallel to the x- and y-axes respectively, p is the pressure, μ the viscosity of the liquid, b the gap between the plates, $\phi = -(b^2/12\mu)p$ is the velocity potential, and the stream function ψ exists by virtue of the equation of continuity. The viscous liquid is taken as extending to $x = +\infty$ and the other fluid to $x = -\infty$. The velocity at infinity is supposed uniform and is denoted by V; hence $\phi \sim Vx$ as $x \to +\infty$.

The pressure is constant in the fluid of negligible viscosity, and if effects due to interfacial stress can be neglected (see (**1**) for a discussion of this assumption), the pressure in the viscous liquid is constant along the interface. The equation of the interface can therefore be taken as $\phi(x, y, t) = 0$. We assume further that the viscous liquid is completely expelled by the other (this assumption is also discussed in (**1**)), so that the interface is

a material surface moving with the mean velocity; hence

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} = 0 \tag{2}$$

on $\phi = 0$.

The position of the interface in the physical plane is not known *a priori*, but it is specified in the potential (ϕ, ψ) plane. We therefore work in the potential plane and consider x and y as functions of ϕ and ψ. The region occupied by the viscous liquid corresponds with the domain $\phi > 0$. Further, ϕ and ψ satisfy the Cauchy–Riemann equations, and therefore $z = x+iy$ is an analytic function of $w = \phi+i\psi$, satisfying

$$z \sim w/V \quad \text{as} \quad \phi \to +\infty, \tag{3}$$

because the velocity at infinity is uniform. The boundary condition on z at the interface is obtained by expressing (2) with ϕ and ψ as independent variables. A straightforward application of the rules for change of variable in partial differentiation gives

$$\frac{\partial x}{\partial t}\frac{\partial y}{\partial \psi} - \frac{\partial x}{\partial \psi}\frac{\partial y}{\partial t} = 1, \tag{4}$$

on $\phi = 0$.

Any function $z(w)$, which is analytic for $\phi > 0$ and satisfies (3) and (4), gives a mathematically possible flow. (4) is non-linear and the problem of finding a solution to fit given initial conditions is in general difficult.

3. Exact solutions representing the growth of fingers

The expression

$$z = \frac{w}{V} + Ut + \frac{2l}{\pi}(1-\lambda)\log \tfrac{1}{2}\left\{1+\exp\left(-\frac{\pi w}{Vl}\right)\right\}, \tag{5}$$

where $V = \lambda U$ and $-Vl \leqslant \psi \leqslant Vl$, was obtained in (**1**) for the steady propagation with velocity U of a semi-infinite finger of asymptotic width $2\lambda l$ in a channel bounded by walls $y = \pm l$ containing viscous liquid moving with velocity V at infinity.† If ψ ranges from $-\infty$ to $+\infty$, (5) represents the motion of equal and equally spaced semi-infinite fingers, each of width $2\lambda l$ and separated from its neighbour by a gap of width $2(1-\lambda)l$, propagating steadily through the viscous liquid. It is clear (as pointed out in (**1**)) that there exists a family of solutions corresponding with values of λ between 0 and 1, and the case of steady motion does not possess a mathematically unique solution.

Whilst searching for a mechanism to distinguish a particular value of

† The term Ut, which is not present in the equivalent expression given in (**1**), arises here because the origin is fixed in space, whereas in (**1**) it was fixed relative to the finger.

λ and make the solution physically unique, it was discovered that

$$z = \frac{w}{V}+d(t)+\frac{2l}{\pi}(1-\lambda)\log\tfrac{1}{2}\left\{1+a(t)\exp\left(-\frac{\pi w}{Vl}\right)\right\} \tag{6}$$

is analytic in $\phi > 0$ if $0 < \lambda < 1$ and $-1 < a(t) < 1$, and satisfies (3) and (4) if

$$(1-\lambda)l\dot{a}+\pi\lambda a\dot{d} = \pi Va,$$

$$2(1-\lambda)(2\lambda-1)la\dot{a}+\pi\{1+(2\lambda-1)a^2\}\dot{d} = \pi V(1+a^2), \tag{7}$$

where the dot denotes differentiation with respect to time. The integrals of (7) are easily found to be

$$\frac{a}{(1-a^2)^{2\lambda(1-\lambda)}} = \alpha e^{\pi Vt/l}, \qquad d = Vt-\frac{2l}{\pi}(1-\lambda)^2\log(1-a^2)+\beta, \tag{8}$$

where α and β are arbitrary real constants.

The interface between the two fluids is the image of $\phi = 0$, and its parametric equation is

$$x = d(t)+\frac{l}{\pi}(1-\lambda)\log\tfrac{1}{4}\left\{1+a^2+2a\cos\left(\frac{\pi\psi}{Vl}\right)\right\},$$

$$y = \frac{\psi}{V}-\frac{2l}{\pi}(1-\lambda)\tan^{-1}\left\{\frac{a\sin(\pi\psi/Vl)}{1+a\cos(\pi\psi/Vl)}\right\}. \tag{9}$$

Examining the shape of the interface as given by (9), it is to be noticed in the first place that there exists a whole family of possible shapes, corresponding with values of λ between 0 and 1, all of which are periodic in y with period $2l$. If $\alpha \ll 1$ and $t \ll (l/V)\log(1/\alpha)$, then the interface has the equation

$$x = Vt+A\cos(\pi y/l)e^{\pi Vt/l}+O(\alpha^2), \tag{10}$$

where $A = (2l/\pi)\alpha(1-\lambda)$. This represents a sinusoidal perturbation with amplitude A of the flat interface in the state of uniform motion, which grows exponentially with amplification factor $\pi Vt/l$. It is to be noticed that the form of the interface is, to the first order in the amplitude, independent of λ. Further, as $t \to \infty$

$$1-a \sim \exp[-\pi Vt/\{2l\lambda(1-\lambda)\}], \qquad d \sim Vt/\lambda, \tag{11}$$

from which it follows that the motion given by (6) tends to that given by (5). Thus, for large t the interface consists of equal and equally spaced long fingers, of width $2\lambda l$ and length of order Vt/λ, propagating into the viscous fluid, the tips of the fingers advancing with velocity V/λ. The solution thus represents the growth of long fingers of arbitrary width from particular, approximately sinusoidal, disturbances of a flat interface.

In Fig. 1 are shown one complete wavelength of the interface for three pairs of values of λ and a. Curve (a) corresponds with $\lambda = 0{\cdot}8$, $a = 0{\cdot}95$;

curve (*b*) with $\lambda = 0{\cdot}2$, $a = 0{\cdot}5$. The amplitudes of the two curves are roughly the same, about $\frac{1}{4}$ of the wavelength, but the flatness of that corresponding to $\lambda = 0{\cdot}8$ is already pronounced. Curve (*c*) shows the interface with $\lambda = 0{\cdot}5$ when the amplitude is somewhat larger; for this curve $a = 0{\cdot}95$. The dotted line shows the position of the interface when there is no disturbance.

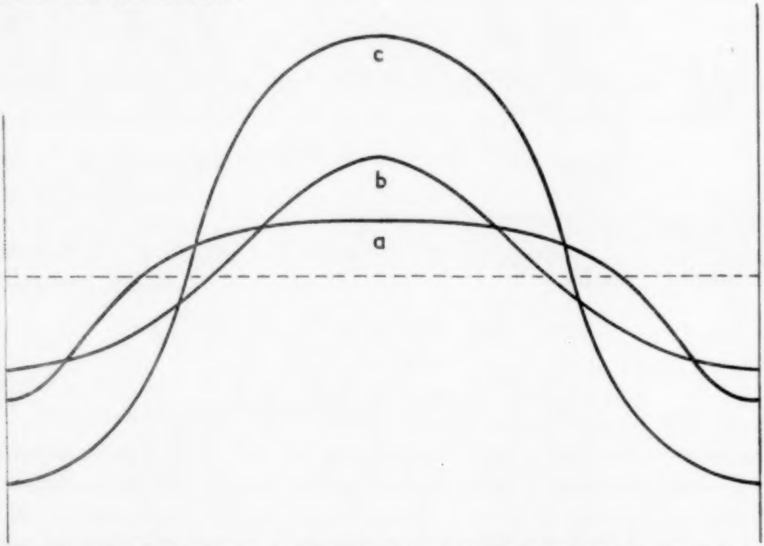

FIG. 1. Shapes of one complete wavelength of the interface for different values of λ. (*a*) $\lambda = 0{\cdot}8$, $a = 0{\cdot}95$; (*b*) $\lambda = 0{\cdot}2$, $a = 0{\cdot}5$; (*c*) $\lambda = 0{\cdot}5$, $a = 0{\cdot}95$.

It was found experimentally in (**1**) that the widths of fingers grown from a flat interface were of the same order as the distance between them. Experiments with single fingers in a channel showed that λ was close to $\frac{1}{2}$ when variations in the pressure drop across the interface due to surface tension are apparently negligible. Now from (11), $a \to 1$ least rapidly or, in other words, the exponential approach to the final asymptotic state is slowest, when $\lambda = \frac{1}{2}$. This distinguishes a particular value of λ but the physical significance, if any, of this result is not clear. No other feature of the solutions which throws light on the occurrence of $\lambda = \frac{1}{2}$ in practice has been detected.

Acknowledgement

The author wishes to thank Sir Geoffrey Taylor for his interest and helpful comments.

REFERENCE

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THE AERODYNAMIC FORCES ON AN AEROFOIL IN UNSTEADY MOTION BETWEEN POROUS WALLS

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SUMMARY

A solution is obtained to the boundary-value problem arising in the unsteady motion of a thin aerofoil in a stream bounded by porous walls. The flow is two-dimensional, inviscid, and incompressible. The condition holding at the porous boundaries is assumed to be a proportionality between the normal velocity component at the wall and the pressure difference across the wall. The solution relies on a form of this boundary condition which is valid for small values of the frequency of the oscillations, although conditions at the aerofoil surface are independent of frequency. The usual assumption of small amplitude of the disturbances is made.

A formula is derived for the pressure on the aerofoil surface in terms of the flow direction at the surface. The relations for the lift and moment about the mid-chord point are obtained for a harmonic upwash distribution. For the case of a rigid aerofoil, particular types of harmonic oscillations are investigated, and the lift and moment are given as dimensionless 'air-load coefficients'. The results are expressed as the first-order terms of expansions in powers of a parameter which depends on the ratio of aerofoil chord to tunnel width, and are valid for small values of this ratio.

1. Introduction

THE problem of an aerofoil moving in a tunnel with porous walls has received comparatively little attention. In a recent paper (**1**), Drake obtained an extended form of Possio's integral equation for an oscillating aerofoil in two-dimensional compressible flow between porous walls. For the steady flow problem, Baldwin, Turner, and Knechtel (**2**) have presented an approximate solution for lift and drag interference in two-dimensional and axially-symmetrical tunnels with both porous and slotted walls. Further solutions for drag interference and the steady lifting aerofoil have been given by Woods (**3**), (**4**).

Particular cases of a porous-wall tunnel are those in which the porosity is zero (solid-wall tunnel) and infinite (free jet). Solutions to these problems have been given, for the unsteady aerofoil, by Rosenblat (**5**) and Woods (**6**) respectively. The present paper is a generalization of these works, and the analysis follows in the main the pattern set by them.

The basic difference between the present problem and the solid-wall and free jet problems lies in the nature of the boundary condition at the walls. For the solid-wall tunnel the component of velocity normal to the walls is constant, while for a free jet the pressure is constant along the

jet boundaries. In the present case it is assumed that the velocity component normal to the wall is proportional to the pressure difference across the wall, where the constant of proportionality is a measure of the porosity. This relationship has been used in their work on porous-wall tunnels by the above-mentioned authors.

It is shown in section 2 below that for oscillations of small amplitude and frequency this condition at the boundary can be expressed as

$$\theta = \varpi\Omega,$$

where θ is the flow direction at the walls, ϖ is the 'porosity' of the walls, Ω is the 'velocity logarithm' $\equiv \ln(U/q)$, q being the fluid velocity and U a standard reference velocity.

The aerofoil is taken to be situated midway between the walls, and the usual assumptions of unsteady aerofoil theory are made. That is, we suppose (i) that the aerofoil is sufficiently thin for the two components of the flow direction at the surface, that is, the symmetrical part associated with the unsteady upwash and the anti-symmetrical part arising from thickness effects, to be additive—the latter part being neglected in the integrations as it makes no contribution to the lift and moment; (ii) that an infinitely thin 'wake' or vortex sheet is generated from the trailing edge; (iii) that the displacements due to unsteadiness are small enough to allow only first-order terms in the perturbation velocity to be taken.

The physical plane of the problem is conformally mapped, by a transformation involving Jacobian elliptic functions, into a rectangle of which one side represents the aerofoil surface, the opposite side the tunnel walls, and the other pair of sides the upper and lower surfaces of the vortex sheet. It is then seen that the function

$$f \equiv \Omega + i\theta = \ln(U/q) + i\theta$$

is analytic everywhere within and on this rectangle, except possibly for a finite number of logarithmic singularities on its border. Moreover f is determined from the boundary conditions (i) known θ at the aerofoil surface, (ii) proportionality of θ and Ω at the walls, and (iii) continuity of pressure across the vortex sheet. An appropriate expression for f in terms of its known boundary values is found in the form of an integral equation.

The requirement that f vanishes at infinity upstream yields an equation relating θ^*, the flow direction at the aerofoil surface, and X, the jump in the velocity logarithm across the wake. A further integral condition connecting these two functions is found from considering the circulation about a closed contour which includes within it the aerofoil and wake. The function X is also seen to satisfy a partial differential equation derived from the fact that the pressure must be continuous across the vortex sheet.

These relations are subsequently used in the determination of the lift and moment.

An expression for the pressure distribution on the aerofoil surface is next derived. By using the above-mentioned relations the pressure is given in a form not involving the function X explicitly. From the pressure, formulae for the lift and moment about the mid-chord point are obtained. Some of the integrals encountered in this determination cannot be evaluated exactly, and it becomes necessary to resort to approximations. It is assumed at this stage that the ratio of aerofoil length to tunnel width is small, and expansions are carried out in terms of a parameter k, which is a direct function of this ratio. Odd powers of k are found to vanish, and formulae for the lift and moment are obtained which are correct to order k^2.

These formulae are in terms of θ^*, the flow direction on the aerofoil surface, which, however, is not completely specified *a priori*. It is shown that θ^* is compounded of a term proportional to the prescribed upwash velocity, together with a term associated with the movement of the front stagnation point. This latter component must be determined separately, from the conditions of the flow.

As a particular case we then consider a rigid body aerofoil performing simultaneous translational and rotational (pitching) oscillations. For this case the lift and moment are expressed as real dimensionless numbers, called air-load coefficients. From these formulae it is possible to evaluate the corrections necessitated by the interference due to the presence of the walls. The author proposes in a future publication to discuss in detail these interference effects for various values of the porosity and of the ratio of aerofoil length to tunnel width.

In the course of the analysis we encounter several integrals involving Jacobian elliptic functions. For the most part no attempt is made here to give the method of evaluating these integrals, and the reader is referred to (**5**), where identical or similar integrals are discussed at some length.

2. The boundary conditions

Consider the unsteady motion of a thin, two-dimensional aerofoil situated midway between straight, parallel, porous walls, as shown in Fig. 1. Let (q, θ) be the velocity vector in polar coordinates of an incompressible flow past the aerofoil. The origin of the $z\ (= x+iy)$ plane is taken to be at the mid-chord point of the aerofoil, whose chord length is $4a$. The width of the tunnel is $2h$, so that the tunnel walls are the lines $y = \pm h$.

It is assumed that the aerofoil is thin enough, and its unsteady perturbations about its mean position are small enough, to allow us to impose the

boundary conditions prevailing on the aerofoil surface over the strip $-2a \leqslant x \leqslant 2a$, $y = 0$ without significant error. It is consequently further assumed that the (infinitely thin) vortex sheet or wake generated from the trailing edge will lie on $y = 0$, $2a \leqslant x < \infty$.

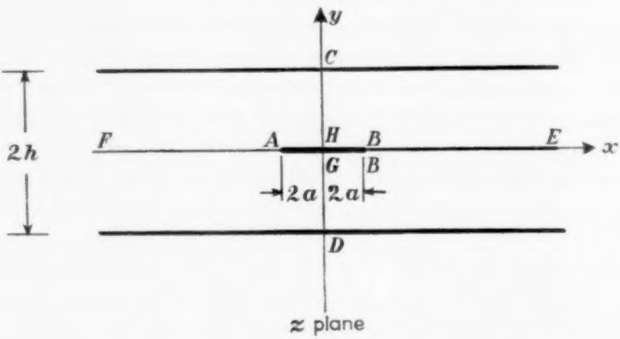

FIG. 1

Suppose now that the pressure outside the porous surfaces is maintained at some constant value, p_s, say. Let v, p denote respectively the component of velocity normal to the walls and the fluid pressure at the walls. Then $p-p_s$ represents the drop in pressure across the porous surface and the basic assumption of the theory is that the normal velocity component is proportional to this pressure difference. We write

$$v = \varpi\frac{(p-p_s)}{\rho U}, \tag{1}$$

where ρ is the density and U the velocity at infinity upstream. The quantity ϖ is a dimensionless parameter, referred to as the 'porosity', whose value is determined experimentally from the character of the surface. It may range between 0 (zero porosity—solid wall) and ∞ (infinite porosity—free jet). The use of the assumption (1) is made in (**1**), (**3**) and elsewhere.

At a point on the wall the flow direction is given by

$$\theta = \sin^{-1}\left(\frac{v}{q}\right),$$

or, to first order in the perturbation velocity $q-U$,

$$\theta = \frac{v}{U}.$$

Hence (1) becomes
$$\theta = \varpi\frac{(p-p_s)}{\rho U^2}. \tag{2}$$

Differentiation, and use of Bernoulli's equation, gives

$$\frac{U^2}{\varpi}\frac{\partial\theta}{\partial s} = \frac{1}{\rho}\frac{\partial p}{\partial s} = -\left[\frac{\partial q}{\partial t}+\frac{\partial}{\partial s}(\tfrac{1}{2}q^2)\right].$$

If now we define a 'velocity logarithm'

$$\Omega \equiv \ln(U/q),$$

then to first order we find

$$\frac{\partial q}{\partial t} = -U\frac{\partial\Omega}{\partial t}, \qquad \frac{\partial}{\partial s}(\tfrac{1}{2}q^2) = -U^2\frac{\partial\Omega}{\partial s},$$

so that we now have

$$\frac{\partial\theta}{\partial s} = \varpi\left[\frac{\partial\Omega}{\partial s}+\frac{1}{U}\frac{\partial\Omega}{\partial t}\right]. \tag{3}$$

In order to render the problem tractable by the present method it is necessary to restrict consideration to the case where $\dfrac{1}{U}\dfrac{\partial\Omega}{\partial t} \ll \dfrac{\partial\Omega}{\partial s}$. For harmonic oscillations we may write

$$\Omega = \Omega_0(s)e^{ivt}, \qquad \theta = \theta_0(s)e^{ivt},$$

so that (3) becomes

$$e^{ivt}\frac{\partial\theta_0}{\partial s} = e^{ivt}\varpi\left[\frac{\partial\Omega_0}{\partial s}+\frac{iv}{U}\Omega_0\right].$$

Hence the solution will be confined to small values of the frequency of oscillation. The boundary condition at the wall may thus be written

$$\theta = \varpi\Omega, \tag{4}$$

where the pressure p_s is selected such that the integration constant vanishes.

3. Solution of Laplace's equation

Suppose $F(t)$ is a function of t $(= \gamma+i\eta)$ which is regular everywhere within and on the rectangle $-2K \leqslant \gamma \leqslant 2K, 0 \leqslant \eta \leqslant K'$ with the possible exception of a finite number of logarithmic singularities on the border of the rectangle. Suppose further that the following data are given concerning the character of $F(t)$:

(i) the imaginary part of $F(t)$ on $\eta = 0$, $-2K \leqslant \gamma \leqslant 2K$ is known, and is equal to $\operatorname{im} F_0$, say;

(ii) the imaginary part of $F(t)$ on $\eta = K'$, $-2K \leqslant \gamma \leqslant 2K$ is known, $= \operatorname{im} F_{K'}$, say;

(iii) for the sides $\gamma = \pm 2K$, $0 \leqslant \eta \leqslant K'$,

$$\operatorname{im} F_{2K} = \operatorname{im} F_{-2K}$$

and $\{\operatorname{re} F_{2K} - \operatorname{re} F_{-2K}\}$ is known.

An expression for $F(t)$ satisfying these conditions has been obtained

(7), and is given in terms of theta functions. A more convenient form of this solution, involving Jacobian elliptic and zeta functions, and derived from it (see Appendix 1) is

$$F(t) =$$
$$A+\frac{1}{2\pi}\int\limits_{-2K}^{2K}\left\{\operatorname{im} F_0\left[\frac{1+\operatorname{cn}(\gamma^*-t)\operatorname{dn}(\gamma^*-t)}{\operatorname{sn}(\gamma^*-t)}+Z(\gamma^*-t)\right]-\right.$$
$$\left.-\operatorname{im} F_{K'}\left[\frac{1+\operatorname{cn}(\gamma^*-t+iK')\operatorname{dn}(\gamma^*-t+iK')}{\operatorname{sn}(\gamma^*-t+iK')}+Z(\gamma^*-t+iK')+\frac{\pi i}{2K}\right]\right\}d\gamma^*+$$
$$+\frac{1}{4\pi}\int\limits_{0}^{K'}(\operatorname{re} F_{2K}-\operatorname{re} F_{-2K})\left[\frac{\operatorname{cn}(i\eta^*-t)\operatorname{dn}(i\eta^*-t)-1}{\operatorname{sn}(i\eta^*-t)}-\right.$$
$$\left.-\frac{\operatorname{cn}(i\eta^*+t)\operatorname{dn}(i\eta^*+t)-1}{\operatorname{sn}(i\eta^*+t)}+Z(i\eta^*-t)-Z(i\eta^*+t)\right]d\eta^*, \quad (5)$$

where
$$A = \frac{1}{4K}\int\limits_{-2K}^{2K}\operatorname{re} F_{K'}\,d\gamma^*,$$

a real constant, and K and K' are the real and imaginary quarter-periods of the elliptic functions.

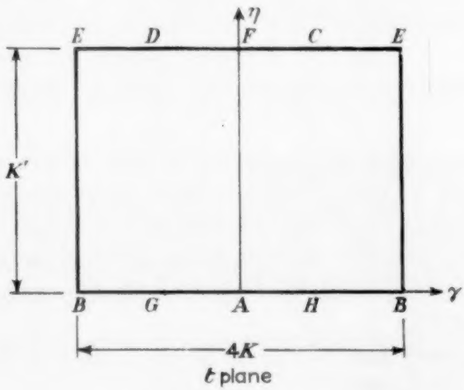

FIG. 2

The physical or z-plane is mapped into a rectangle in the $t = \gamma+i\eta$ plane, shown in Fig. 2, by means of the conformal transformation

$$\operatorname{cn}(t, k) = -\frac{k'}{k}\sinh\frac{\pi z}{2h}, \quad (6)$$

where k and k' are the moduli of the elliptic functions and are given by

$$\left.\begin{aligned} k &= \tanh\frac{\pi a}{h} \\ \text{or, since } k^2+k'^2 = 1, \qquad \frac{k}{k'} &= \sinh\frac{\pi a}{h} \end{aligned}\right\}. \tag{7}$$

In the t-plane the aerofoil surface $y = 0$, $-2a \leqslant x \leqslant 2a$ maps into the line $\eta = 0$, $-2K \leqslant \gamma \leqslant 2K$, the upper and lower surfaces of the vortex sheet map into $\gamma = +2K$, $0 \leqslant \eta \leqslant K'$ and $\gamma = -2K$, $0 \leqslant \eta \leqslant K'$ respectively, while the upper and lower walls become $\eta = K'$, $0 \leqslant \gamma \leqslant 2K$ and $\eta = K'$, $-2K \leqslant \gamma \leqslant 0$ respectively. The point upstream at infinity, F, becomes $t = iK'$, while the point downstream at infinity, E, becomes $t = \pm 2K + iK'$.

If now we define a function $f(t)$ by

$$f \equiv \ln(U/q)+i\theta = \Omega+i\theta, \tag{8}$$

then f will be an analytic function of both z and t, except at possible logarithmic singularities corresponding to stagnation points on the aerofoil surface. Moreover it is clear that $f(t)$ satisfies conditions (i) and (iii) above. For on $\eta = 0$, $-2K \leqslant \gamma \leqslant 2K$ the imaginary part of f is the prescribed flow direction at the aerofoil surface, θ^*, say; while for the sides $\gamma = \pm 2K$, $0 \leqslant \eta \leqslant K'$, we have that the flow direction is the same on either side of the wake, and that the jump in Ω, which we write as

$$X \equiv \Omega_{2K}-\Omega_{-2K}, \tag{9}$$

is determined from the condition of pressure continuity across the vortex sheet.

The conditions on $f(t)$ may be summarized as follows:

$$\left.\begin{array}{l} \text{On } \eta = 0 \text{ and } \eta = K',\ -2K \leqslant \gamma \leqslant 2K \\ \qquad\qquad\qquad \theta-\varpi\Omega = \psi(\gamma) \\ \text{where on} \\ \quad \eta = 0, \quad -2K \leqslant \gamma \leqslant 2K \qquad \varpi = 0 \quad \text{and} \quad \psi = \theta^* \\ \quad \eta = K', \ -2K \leqslant \gamma \leqslant 0 \qquad \varpi = -\varpi \quad \text{and} \quad \psi = 0 \\ \quad \eta = K', \quad\ \ 0 \leqslant \gamma \leqslant 2K \qquad \varpi = \varpi \quad \text{and} \quad \psi = 0 \end{array}\right\}. \tag{10}$$

$$\left.\begin{array}{l} \text{On } \gamma = \pm 2K,\ 0 \leqslant \eta \leqslant K' \\ \qquad\qquad\qquad \theta_{2K} = \theta_{-2K} \\ \qquad\qquad\qquad \Omega_{2K}-\Omega_{-2K} = X \end{array}\right\}. \tag{11}$$

It follows from equation (10) that a solution for $f(t)$ cannot be obtained by direct application of the formula (5), since $\operatorname{im} f_{K'}$ is not known. However $f(t)$ is derived by the following method which utilizes equation (5).

Let $f_m = \Omega_m + i\theta_m$ be an analytic function of t, regular in

$$-2K \leqslant \gamma \leqslant 2K, \qquad 0 \leqslant \eta \leqslant K',$$

which satisfies the 'homogeneous' boundary conditions:

$$\left.\begin{array}{l}
\text{On } \eta = 0 \text{ and } \eta = K',\ -2K \leqslant \gamma \leqslant 2K \\
\qquad\qquad\qquad \theta_m - \varpi\Omega_m = 0 \\
\text{where on} \\
\quad \eta = 0, \quad -2K \leqslant \gamma \leqslant 2K \qquad \varpi = 0 \quad \text{and} \quad \theta_m = 0 \\
\quad \eta = K', \ -2K \leqslant \gamma \leqslant 0 \qquad \varpi = -\varpi \quad \text{and} \quad \theta_m = -\varpi\Omega_m \\
\quad \eta = K', \quad 0 \leqslant \gamma \leqslant 2K \qquad \varpi = \varpi \quad \text{and} \quad \theta_m = \varpi\Omega_m
\end{array}\right\}. \tag{12}$$

$$\left.\begin{array}{l}
\text{On } \gamma = \pm 2K,\ 0 \leqslant \eta \leqslant K', \\
\qquad \theta_m(2K+i\eta) = \theta_m(-2K+i\eta) = 0 \\
\qquad \Omega_m(2K+i\eta) = \Omega_m(-2K+i\eta) = \Omega'_m, \quad \text{say}
\end{array}\right\}. \tag{13}$$

We then show that the function defined by

$$\Phi = f/f_m \tag{14}$$

satisfies all the conditions for the application of the solution (5). From its definition, $\Phi(t)$ can be written

$$\Phi = \frac{(\varpi+i)(\Omega+i\theta)}{(\varpi+i)(\Omega_m+i\theta_m)} = \frac{\Omega+\varpi\theta}{\Omega_m+\varpi\theta_m} + i\,\frac{\theta-\varpi\Omega}{\Omega_m+\varpi\theta_m}, \tag{15}$$

on using (12). From this and equation (10) it is seen that the imaginary parts of Φ on $\eta = 0$ and $\eta = K'$ are given respectively by

$$\operatorname{im}\Phi_0 = \theta^*/(\Omega_m)_0, \qquad \operatorname{im}\Phi_{K'} = 0; \tag{16}$$

while from (11) and (13) we have

$$\operatorname{re}(\Phi_{2K} - \Phi_{-2K}) = X/\Omega'_m. \tag{17}$$

Equations (16) and (17) show that the solution (15) is applicable to Φ once the function f_m is determined.

To obtain f_m, consider the function

$$\ln f_m = \tfrac{1}{2}\ln(\Omega_m^2+\theta_m^2) + i\tan^{-1}\!\left(\frac{\theta_m}{\Omega_m}\right),$$

an analytic function of t. From (12) and (13) the boundary conditions for $\ln f_m$ are:

$$\begin{array}{lll}
\text{on} & \eta = 0,\ -2K \leqslant \gamma \leqslant 2K & \operatorname{im}\ln f_m = 0, \\
& \eta = K',\ -2K \leqslant \gamma \leqslant 0 & \operatorname{im}\ln f_m = -\tan^{-1}\varpi, \\
& \eta = K',\ 0 \leqslant \gamma \leqslant 2K & \operatorname{im}\ln f_m = \tan^{-1}\varpi; \\
\text{on} & \gamma = \pm 2K,\ 0 \leqslant \eta \leqslant K' & \operatorname{re}[(\ln f_m)_{2K} - (\ln f_m)_{-2K}] = 0.
\end{array}$$

Hence a solution may be obtained from equation (5). We find

$$\ln f_m(t) = A_m + \frac{1}{2\pi}\tan^{-1}\varpi \int\limits_{-2K}^{0} \left[\frac{1+\mathrm{cn}(\gamma^*-t+iK')\mathrm{dn}(\gamma^*-t+iK')}{\mathrm{sn}(\gamma^*-t+iK')} + \right.$$

$$\left. + Z(\gamma^*-t+iK') + \frac{\pi i}{2K}\right] d\gamma^* - \frac{1}{2\pi}\tan^{-1}\varpi \times$$

$$\times \int\limits_{0}^{2K} \left[\frac{1+\mathrm{cn}(\gamma^*-t+iK')\mathrm{dn}(\gamma^*-t+iK')}{\mathrm{sn}(\gamma^*-t+iK')} + Z(\gamma^*-t+iK') + \frac{\pi i}{2K}\right] d\gamma^*.$$

After performing the necessary integrations we obtain

$$\ln f_m = A_m + \frac{1}{\pi}\tan^{-1}\varpi \ln\left(\frac{1-k\,\mathrm{cd}\,t}{1+k\,\mathrm{cd}\,t}\right). \tag{18}$$

If we define a quantity ϵ by

$$\tan^{-1}\varpi = \pi\epsilon, \tag{19}$$

then (18) gives

$$f_m = A_1\left(\frac{1-k\,\mathrm{cd}\,t}{1+k\,\mathrm{cd}\,t}\right)^{\epsilon}, \tag{20}$$

where $A_1 = \exp A_m$.

We are now in a position to calculate the values of f_m on the boundaries, and hence, from (16) and (17), a formula for $\Phi(t)$. On

$$\eta = 0,\ -2K \leqslant \gamma \leqslant 2K,$$

we have $\theta_m = 0$, and so from (20),

$$\Omega_m = A_1\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)^{\epsilon}.$$

Thus by (16),

$$\mathrm{im}\,\Phi_0 = \frac{1}{A_1}\theta^*\left(\frac{1+k\,\mathrm{cd}\,\gamma}{1-k\,\mathrm{cd}\,\gamma}\right)^{\epsilon}. \tag{21}$$

Similarly on $\gamma = \pm 2K$, $0 \leqslant \eta \leqslant K'$, we have

$$\Omega'_m = \mathrm{re}\,f_m(\pm 2K+i\eta) = A_1\left(\frac{1+k\,\mathrm{cd}\,i\eta}{1-k\,\mathrm{cd}\,i\eta}\right)^{\epsilon},$$

so that by (17),

$$\mathrm{re}(\Phi_{2K}-\Phi_{-2K}) = \frac{1}{A_1}X\left(\frac{1-k\,\mathrm{cd}\,i\eta}{1+k\,\mathrm{cd}\,i\eta}\right)^{\epsilon}. \tag{22}$$

Using equations (21) and (22), together with $\mathrm{im}\,\Phi_{K'} = 0$, we find from (5) that a solution for Φ is

$$\Phi(t) =$$

$$A + \frac{1}{2\pi A_1}\left\{\int\limits_{-2K}^{2K} \theta^*\left(\frac{1+k\,\mathrm{cd}\,\gamma^*}{1-k\,\mathrm{cd}\,\gamma^*}\right)^{\epsilon}\left[\frac{1+\mathrm{cn}(\gamma^*-t)\mathrm{dn}(\gamma^*-t)}{\mathrm{sn}(\gamma^*-t)} + Z(\gamma^*-t)\right] d\gamma^* + \right.$$

$$+ \int\limits_{0}^{K'} X\left(\frac{1-k\,\mathrm{cd}\,i\eta^*}{1+k\,\mathrm{cd}\,i\eta^*}\right)^{\epsilon}\left[\frac{\mathrm{cn}(i\eta^*-t)\mathrm{dn}(i\eta^*-t)-1}{\mathrm{sn}(i\eta^*-t)} - \right.$$

$$\left.\left. - \frac{\mathrm{cn}(i\eta^*+t)\mathrm{dn}(i\eta^*+t)-1}{\mathrm{sn}(i\eta^*+t)} + Z(i\eta^*-t) - Z(i\eta^*+t)\right] d\eta^*\right\}. \tag{23}$$

This equation, together with (14) and (20), immediately yields an expression for $f(t)$. After rearranging the terms with the aid of the addition properties of the elliptic functions we obtain

$$f(t) = \left(\frac{1-k\,\mathrm{cd}\,t}{1+k\,\mathrm{cd}\,t}\right)^{\epsilon}\left\{A_f+\frac{1}{2\pi}\int\limits_{-2K}^{2K}\theta^*\left(\frac{1+k\,\mathrm{cd}\,\gamma^*}{1-k\,\mathrm{cd}\,\gamma^*}\right)^{\epsilon}\left[\frac{\mathrm{cn}\,\gamma^*\,\mathrm{dn}\,\gamma^*+\mathrm{cn}\,t\,\mathrm{dn}\,t}{\mathrm{sn}\,\gamma^*-\mathrm{sn}\,t}-\right.\right.$$
$$\left.-Z(t)+Z(\gamma^*)\right]d\gamma^*+$$
$$\left.+\frac{1}{2\pi}\int\limits_{0}^{K'}X\left(\frac{1-k\,\mathrm{cd}\,i\eta^*}{1+k\,\mathrm{cd}\,i\eta^*}\right)^{\epsilon}\left[\mathrm{sn}\,t\,\frac{\mathrm{cn}\,t\,\mathrm{dn}\,t-\mathrm{cn}\,i\eta^*\,\mathrm{dn}\,i\eta^*}{\mathrm{sn}^2 i\eta^*-\mathrm{sn}^2 t}-Z(t)\right]d\eta^*\right\} \quad (24)$$

(where $A_f = AA_m$).

4. Conditions on the function $f(t)$

We first apply the condition that the flow upstream is undisturbed. Taking the limit $t \to iK'$ in equation (24), and equating real and imaginary parts, we find

$$A_f+\frac{1}{2\pi}\int\limits_{-2K}^{2K}\theta^*\left(\frac{1+k\,\mathrm{cd}\,\gamma^*}{1-k\,\mathrm{cd}\,\gamma^*}\right)^{\epsilon}[-k\,\mathrm{sn}\,\gamma^*+Z(\gamma^*)]\,d\gamma^* = 0 \qquad (25)$$

and
$$\int\limits_{-2K}^{2K}\theta^*\left(\frac{1+k\,\mathrm{cd}\,\gamma^*}{1-k\,\mathrm{cd}\,\gamma^*}\right)^{\epsilon}d\gamma^*+\int\limits_{0}^{K'}X\left(\frac{1-k\,\mathrm{cd}\,i\eta^*}{1+k\,\mathrm{cd}\,i\eta^*}\right)^{\epsilon}d\eta^* = 0. \qquad (26)$$

With the aid of these equations (24) can be written

$$f(t)=$$
$$\frac{1}{2\pi}\left(\frac{1-k\,\mathrm{cd}\,t}{1+k\,\mathrm{cd}\,t}\right)^{\epsilon}\left\{\int\limits_{-2K}^{2K}\theta^*\left(\frac{1+k\,\mathrm{cd}\,\gamma^*}{1-k\,\mathrm{cd}\,\gamma^*}\right)^{\epsilon}\left[\frac{\mathrm{cn}\,\gamma^*\,\mathrm{dn}\,\gamma^*+\mathrm{cn}\,t\,\mathrm{dn}\,t}{\mathrm{sn}\,\gamma^*-\mathrm{sn}\,t}+k\,\mathrm{sn}\,\gamma^*\right]d\gamma^*+\right.$$
$$\left.+\int\limits_{0}^{K'}X\left(\frac{1-k\,\mathrm{cd}\,i\eta^*}{1+k\,\mathrm{cd}\,i\eta^*}\right)^{\epsilon}\mathrm{sn}\,t\,\frac{\mathrm{cn}\,t\,\mathrm{dn}\,t-\mathrm{cn}\,i\eta^*\,\mathrm{dn}\,i\eta^*}{\mathrm{sn}^2 i\eta^*-\mathrm{sn}^2 t}\,d\eta^*\right\}. \qquad (27)$$

Further conditions on $f(t)$ may be obtained by considering the circulation. If the unsteady motion is assumed to start from rest at time $t = 0$, then the circulation about the aerofoil is initially zero. At any subsequent time t the circulation about any contour enclosing the aerofoil and (finite) wake will, by Kelvin's theorem, also be zero. In particular, if the unsteady motion has persisted for an indefinitely long time, the contour will extend downstream to infinity. Hence we have

$$\int\limits_C dw = 0,$$

where w is the complex potential function, and C is the closed circuit

containing the aerofoil and vortex sheet, shown in Fig. 3 (*a*). Since

$$\frac{dw}{dz} = \frac{q}{U}e^{-i\theta} = e^{-f},$$

therefore
$$\int\limits_C e^{-f}\,dz = 0.$$

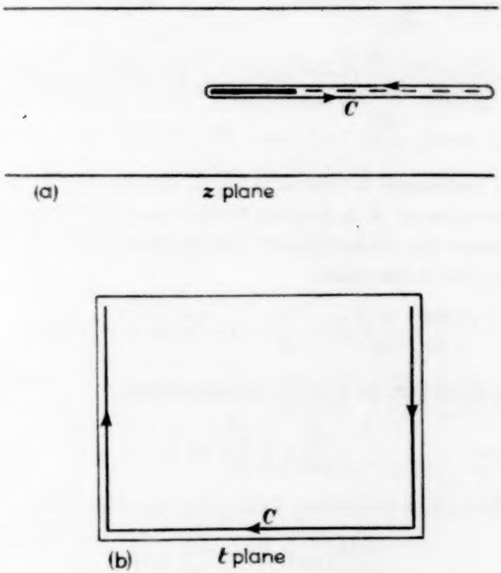

FIG. 3

To first order in the perturbations the last equation may be written

$$\int\limits_C f(z)\,dz = 0,$$

since $\int\limits_C dz = 0$. Differentiation of equation (6) gives

$$dz = \frac{\pi}{2hk}\operatorname{sn} t\,dt, \tag{28}$$

so that we have
$$\int\limits_C f(t)\operatorname{sn} t\,dt = 0,$$

where C is now the path shown in Fig. 3 (*b*). Equating real and imaginary parts of this equation gives

$$\int\limits_{-2K}^{2K} \theta^*\operatorname{sn}\gamma\,d\gamma = 0 \tag{29}$$

and
$$\int\limits_{-2K}^{2K} \Omega(\gamma)\operatorname{sn}\gamma\,d\gamma - i\int\limits_0^{K'} X\operatorname{sn} i\eta\,d\eta = 0. \tag{30}$$

The form of equation (30) is unsatisfactory as it contains the unknown function $\Omega(\gamma)$. Hence we substitute the appropriate limits from (27) into this equation and express it as

$$\int_{-2K}^{2K}\theta^*\left(\frac{1+k\operatorname{cd}\gamma^*}{1-k\operatorname{cd}\gamma^*}\right)^{\epsilon}\int_{-2K}^{2K}\left(\frac{1-k\operatorname{cd}\gamma}{1+k\operatorname{cd}\gamma}\right)^{\epsilon}\operatorname{sn}\gamma\frac{\operatorname{cn}\gamma\operatorname{dn}\gamma+\operatorname{cn}\gamma^*\operatorname{dn}\gamma^*}{\operatorname{sn}\gamma^*-\operatorname{sn}\gamma}\,d\gamma d\gamma^*+$$
$$+\int_{0}^{K'}X\left(\frac{1-k\operatorname{cd}i\eta^*}{1+k\operatorname{cd}i\eta^*}\right)^{\epsilon}\int_{-2K}^{2K}\left(\frac{1-k\operatorname{cd}\gamma}{1+k\operatorname{cd}\gamma}\right)^{\epsilon}\operatorname{sn}^2\gamma\frac{\operatorname{cn}\gamma\operatorname{dn}\gamma-\operatorname{cn}i\eta^*\operatorname{dn}i\eta^*}{\operatorname{sn}^2 i\eta^*-\operatorname{sn}^2\gamma}\,d\gamma d\eta^*$$
$$=0. \quad (31)$$

This equation represents a further integral condition relating θ^* and X.

Finally, the value of X is subject to the condition that the pressure is continuous across the vortex sheet. From Bernoulli's equation we have that on either side of the sheet

$$\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}(\tfrac{1}{2}q^2)+\frac{\partial q}{\partial t}=0 \quad (y=0,\ x>2a),$$

so that across the sheet we obtain, to first order,

$$\frac{\partial X}{\partial x}+\frac{1}{U}\frac{\partial X}{\partial t}=0. \quad (32)$$

In terms of the t-plane variables, this is, by equation (28),

$$\frac{\pi U}{2hk}i\operatorname{ns}i\eta\frac{\partial X}{\partial \eta}+\frac{\partial X}{\partial t}=0. \quad (33)$$

5. Pressure on the aerofoil surface

For an upwash of general form, an exact expression may now be derived for the pressure distribution on the aerofoil surface. Bernoulli's equation can be written

$$p=\tfrac{1}{2}\rho(U^2-q^2)-\rho\frac{\partial}{\partial t}\int q\,ds, \quad (34)$$

the constant being omitted as it makes no contribution to the lift and moment. Writing $s=x$ on the aerofoil surface, we obtain from equation (28)

$$ds=dx=\frac{2hk}{\pi}\operatorname{sn}\gamma\,d\gamma,$$

and so to first order in the perturbation velocity $q-U$, (34) can be written

$$p=\rho U^2\Omega+\frac{2hk\rho U}{\pi}\int_0^{\gamma}\operatorname{sn}\gamma\frac{\partial\Omega}{\partial t}\,d\gamma. \quad (35)$$

Ω is the velocity logarithm on the aerofoil surface, and from equation (27) is

$$\Omega = \frac{1}{2\pi}\int\limits_{-2K}^{2K} \theta^*\left(\frac{1+k\,\mathrm{cd}\,\gamma^*}{1-k\,\mathrm{cd}\,\gamma^*}\right)^{\epsilon}\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)^{\epsilon}\left\{\frac{\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma+\mathrm{cn}\,\gamma^*\,\mathrm{dn}\,\gamma^*}{\mathrm{sn}\,\gamma^*-\mathrm{sn}\,\gamma}+k\,\mathrm{sn}\,\gamma^*\right\}d\gamma^*+$$

$$+\frac{1}{2\pi}\int\limits_{0}^{K'} X\left(\frac{1-k\,\mathrm{cd}\,i\eta^*}{1+k\,\mathrm{cd}\,i\eta^*}\right)^{\epsilon}\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)^{\epsilon}\mathrm{sn}\,\gamma\,\frac{\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma-\mathrm{cn}\,i\eta^*\,\mathrm{dn}\,i\eta^*}{\mathrm{sn}^2 i\eta^*-\mathrm{sn}^2\gamma}\,d\eta^*. \quad (36)$$

We now substitute for Ω from (36) into (35). It is shown in Appendix 2 that an expression for the pressure distribution is

$$p = \frac{\rho U}{2\pi}\int\limits_{-2K}^{2K}(U\theta^*+\dot{G}^*)\left(\frac{1+k\,\mathrm{cd}\,\gamma^*}{1-k\,\mathrm{cd}\,\gamma^*}\right)^{\epsilon}\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)^{\epsilon}\times$$

$$\times\left[\frac{\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma+\mathrm{cn}\,\gamma^*\,\mathrm{dn}\,\gamma^*}{\mathrm{sn}\,\gamma^*-\mathrm{sn}\,\gamma}-Z(\gamma)+k\,\mathrm{sn}\,\gamma^*\right]d\gamma^*+$$

$$+\frac{\rho U\epsilon k}{\pi}\int\limits_{-2K}^{2K}(U\theta^*+\dot{G}^*)\left(\frac{1+k\,\mathrm{cd}\,\gamma^*}{1-k\,\mathrm{cd}\,\gamma^*}\right)^{\epsilon}\int\limits_{0}^{\gamma}\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)^{\epsilon}\times$$

$$\times\left[\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma+\mathrm{sn}\,\gamma Z(\gamma)-\frac{H}{F}\right]d\gamma d\gamma^*, \quad (37)$$

where we have put
$$G(\gamma^*) \equiv \frac{2hk}{\pi}\int\limits_{0}^{\gamma^*}\theta^*\,\mathrm{sn}\,\gamma\,d\gamma, \quad (38)$$

and where F, H are constants defined by

$$F \equiv \int\limits_{-2K}^{2K}\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)^{\epsilon}d\gamma \quad (39)$$

and
$$H \equiv \int\limits_{-2K}^{2K}\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)^{\epsilon}[\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma+\mathrm{sn}\,\gamma Z(\gamma)]\,d\gamma. \quad (40)$$

It is an important feature that equation (37) for p contains no terms involving X, and so the expressions for the lift and moment will similarly be free of such terms.

6. Movement of the front stagnation point

It is clear from equation (37) that the pressure, and hence the lift and moment, are determined once the flow direction at the aerofoil surface, θ^*, is given. However θ^* is not completely known *a priori*, since it is composed not only of a term arising from the prescribed upwash velocity, but also of a term associated with the motion of the front stagnation point.

The latter contribution is not prescribed, and must be calculated from the conditions of the flow.

The position of the front stagnation point in the mean steady flow is at $\gamma^* = 0$ in the t plane. Suppose that at any instant of the unsteady motion its position is at $\gamma^* = -\delta$, where δ will be small, since the amplitude of the unsteady motion is small. Then in the range $-\delta \leqslant \gamma^* \leqslant 0$, the flow direction will be reversed, that is, θ^* will be increased by π in this range.

If $v(\gamma^*, t)$ be the upwash distribution, then apart from the above term the flow direction at the aerofoil is $\theta^* = v/U$ to first order. Taking a harmonic upwash of the form

$$v(\gamma, t) = v_0(\gamma)e^{i\nu t},$$

then at the aerofoil surface we may write

$$\theta^* = \begin{cases} \dfrac{v_0}{U}e^{i\nu t} & (-2K \leqslant \gamma \leqslant 2K) \\ \pi & (-\delta \leqslant \gamma \leqslant 0). \end{cases} \tag{41}$$

Moreover equations (32) and (33) enable us to take X in the form

$$X = X_0 e^{i\nu(t-x/U)} = X_0 e^{i\nu t}\left(\frac{1-k\,\mathrm{cd}\,i\eta}{1+k\,\mathrm{cd}\,i\eta}\right)^{i\nu h/\pi U}, \tag{42}$$

where X_0 is a constant.

The quantity δ can be uniquely determined from equations (26) and (31). It is found convenient to replace (31) by a linear combination of it with (26), namely,

$$\int\limits_{-2K}^{2K} \theta^*\left(\frac{1+k\,\mathrm{cd}\,\gamma^*}{1-k\,\mathrm{cd}\,\gamma^*}\right)^{\epsilon} \int\limits_{-2K}^{2K} \left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)^{\epsilon}\left\{\mathrm{sn}\,\gamma\,\frac{\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma+\mathrm{cn}\,\gamma^*\,\mathrm{dn}\,\gamma^*}{\mathrm{sn}\,\gamma^*-\mathrm{sn}\,\gamma}+\right.$$
$$\left.+\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma\right\}d\gamma d\gamma^*+$$
$$+\int\limits_{0}^{K'} X\left(\frac{1-k\,\mathrm{cd}\,i\eta^*}{1+k\,\mathrm{cd}\,i\eta^*}\right)^{\epsilon} \int\limits_{-2K}^{2K} \left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)^{\epsilon}\times$$
$$\times\left\{\mathrm{sn}^2\gamma\,\frac{\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma-\mathrm{cn}\,i\eta^*\,\mathrm{dn}\,i\eta^*}{\mathrm{sn}^2 i\eta^*-\mathrm{sn}^2\gamma}+\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma\right\}d\gamma d\eta^* = 0. \tag{43}$$

To evaluate the integrals occurring here, as well as the constants F and H, it is necessary to expand the integrands in series and integrate term by term. Since thickness effects are being neglected, we may assume our results to be valid for small values of the ratio aerofoil length to tunnel width. Hence we expand in powers of the elliptic parameter k, defined by equation (7) to be a direct function of this ratio. The basic approximations

are found to be

$$\left(\frac{1-k\,\mathrm{cd}\,u}{1+k\,\mathrm{cd}\,u}\right)^{\epsilon} = 1-2\epsilon k\,\mathrm{cd}\,u+2\epsilon^2k^2\,\mathrm{cd}^2u+O(k^3) \tag{44}$$

and
$$\left(\frac{1+k\,\mathrm{cd}\,u}{1-k\,\mathrm{cd}\,u}\right)^{\epsilon} = 1+2\epsilon k\,\mathrm{cd}\,u+2\epsilon^2k^2\,\mathrm{cd}^2u+O(k^3). \tag{45}$$

The formulae (41) and (42) for θ^* and X are now substituted into equations (26) and (41). Using (44) and (45) we carry out the integrations and obtain, to first order in δ,

$$\pi\delta\left(\frac{1+k}{1-k}\right)^{\epsilon}+(1+2\epsilon^2k^2)\frac{e^{i\nu t}}{U}\int\limits_{-2K}^{2K} v_0(\gamma^*)[1+2\epsilon k\,\mathrm{cn}\,\gamma^*-2\epsilon^2k^2\,\mathrm{sn}^2\gamma^*]\,d\gamma^*+$$
$$+X_0e^{i\nu t}\int\limits_0^{K'}\left(\frac{1-k\,\mathrm{cd}\,i\eta}{1+k\,\mathrm{cd}\,i\eta}\right)^{\epsilon+(i\nu h/\pi U)}d\eta = 0 \tag{46}$$

and

$$\pi\delta\left(\frac{1+k}{1-k}\right)^{\epsilon}+(1+2\epsilon^2k^2)\frac{e^{i\nu t}}{U}\int\limits_{-2K}^{2K} v_0(\gamma^*)[\mathrm{cn}\,\gamma^*\,\mathrm{dn}\,\gamma^*+\mathrm{sn}\,\gamma^*\,Z(\gamma^*)+$$
$$+2\epsilon k(1-2\epsilon^2k^2\,\mathrm{sn}^2\gamma^*)]\,d\gamma^*-$$
$$-\frac{1}{F}X_0e^{i\nu t}\int\limits_0^{K'}\left(\frac{1-k\,\mathrm{cd}\,i\eta}{1+k\,\mathrm{cd}\,i\eta}\right)^{\epsilon+(i\nu h/\pi U)}\times$$
$$\times\int\limits_{-2K}^{2K}\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)^{\epsilon}\left[\mathrm{sn}^2\gamma\,\frac{\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma-\mathrm{cn}\,i\eta^*\,\mathrm{dn}\,i\eta^*}{\mathrm{sn}^2i\eta^*-\mathrm{sn}^2\gamma}+\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma\right]d\gamma d\eta^* = 0. \tag{47}$$

By elimination of X_0, δ may be obtained from these equations.

The integrals involving η may be evaluated exactly in terms of Legendre functions. It is shown in Appendix 3 that

$$\int\limits_0^{K'}\left(\frac{1-k\,\mathrm{cd}\,i\eta}{1+k\,\mathrm{cd}\,i\eta}\right)^{\epsilon+(i\nu h/\pi U)}\int\limits_{-2K}^{2K}\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)^{\epsilon}\times$$
$$\times\left\{\mathrm{sn}^2\gamma\,\frac{\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma-\mathrm{cn}\,i\eta^*\,\mathrm{dn}\,i\eta^*}{\mathrm{sn}^2i\eta^*-\mathrm{sn}^2\gamma}+\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma\right\}d\gamma d\eta^*$$
$$=\frac{i\lambda+\epsilon r}{i\lambda}F\int\limits_0^{K'}\left(\frac{1-k\,\mathrm{cd}\,i\eta}{1+k\,\mathrm{cd}\,i\eta}\right)^{\epsilon+(i\lambda/r)}[\mathrm{cn}\,i\eta\,\mathrm{dn}\,i\eta+\mathrm{sn}\,i\eta\,Z(i\eta)]\,d\eta-$$
$$-\frac{\epsilon r}{i\lambda}H\int\limits_0^{K'}\left(\frac{1-k\,\mathrm{cd}\,i\eta}{1+k\,\mathrm{cd}\,i\eta}\right)^{\epsilon+(i\lambda/r)}d\eta,$$

where we have put
$$\frac{2a\nu}{U}=\lambda,\qquad \frac{2\pi a}{h}=r. \tag{48}$$

Transforming to the variable ξ, defined by

$$\frac{x}{2a} = \xi,$$

we find

$$I_1 \equiv \int_0^{K'} \left(\frac{1-k\,\mathrm{cd}\,i\eta}{1+k\,\mathrm{cd}\,i\eta}\right)^{\epsilon+(i\lambda/r)} d\eta = \frac{r}{\sqrt{2k'}} \int_1^{\infty} \frac{e^{-\xi(\epsilon r+i\lambda)}\,d\xi}{\sqrt{\{\cosh r\xi - \cosh r\}}},$$

and

$$I_2 \equiv \int_0^{K'} \left(\frac{1-k\,\mathrm{cd}\,i\eta}{1+k\,\mathrm{cd}\,i\eta}\right)^{\epsilon+(i\lambda/r)} [\mathrm{cn}\,i\eta\,\mathrm{dn}\,i\eta + \mathrm{sn}\,i\eta\, Z(i\eta)]\,d\eta$$

$$= \frac{rk'}{2\sqrt{2k}} \int_1^{\infty} e^{-\xi(\epsilon r+i\lambda)} \left\{\frac{\sinh r\xi}{\sqrt{\{\cosh r\xi - \cosh r\}}} + \frac{\sqrt{2}i}{k'} Z\left[\mathrm{cn}^{-1}\left(\frac{k'}{k}\sinh\frac{r\xi}{2}\right)\right]\right\} d\xi.$$

It has been shown (**5**) that these integrals are expressible as

$$I_1 = \frac{1}{k'} Q_N(\cosh r)$$

and

$$I_2 = \frac{rk'}{4k(i\lambda+\epsilon r)}\left[NQ_{N-1}(\cosh r) + \left(\frac{2E}{Kk'^2} - 1\right)Q_N(\cosh r) - (N+1)Q_N(\cosh r)\right],$$

where $Q_N(z)$ is the Legendre function of the second kind, E is the complete elliptic integral of the second kind, and

$$N \equiv \frac{i\lambda}{r} + \epsilon - \tfrac{1}{2}. \tag{49}$$

If for brevity we define

$$a_0 \equiv \frac{1}{\pi U} \int_{-2K}^{2K} v_0[1 - 2\epsilon^2 k^2 \mathrm{sn}^2\gamma]\,d\gamma \tag{50}$$

and

$$b_0 \equiv \frac{1}{\pi U} \int_{-2K}^{2K} v_0[\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma + \mathrm{sn}\,\gamma\, Z(\gamma)]\,d\gamma, \tag{51}$$

then equations (46) and (47) may now be written, correct to order k^2,

$$\delta\left(\frac{1+k}{1-k}\right)^{\epsilon} + (1+2\epsilon^2k^2)e^{i\nu t}(a_0 + 2\epsilon k b_0) + \frac{X_0 e^{i\nu t}}{\pi}\cdot\frac{1}{k'} Q_N = 0 \tag{52}$$

and

$$\delta\left(\frac{1+k}{1-k}\right)^{\epsilon} + (1+2\epsilon^2k^2)e^{i\nu t}(b_0 + 2\epsilon k a_0) -$$

$$-\frac{X_0 e^{i\nu t}}{\pi}\,\frac{r}{i\lambda}\left\{\frac{k'}{4k}\left[NQ_{N-1} + \left(\frac{2E}{Kk'^2} - 1\right)Q_N - (N+1)Q_{N+1}\right] - \frac{\epsilon H}{k'F} Q_N\right\} = 0. \tag{53}$$

Eliminating X_0 from (52) and (53) yields, to the required order of accuracy,

$$\delta = -e^{i\nu t}\{\tfrac{1}{2}(a_0+b_0) + \tfrac{1}{2}(a_0-b_0)T(\lambda, r, \epsilon)\}, \tag{54}$$

where

$$T(\lambda, r, \epsilon) \equiv \frac{1-2\epsilon k}{1+2\epsilon k} \frac{NQ_{N-1}-(N+1)Q_{N+1}+\left(\dfrac{2E}{KK'^2}-1-\dfrac{4\epsilon kH}{Fk'^2}-\dfrac{4i\lambda k}{rk'^2}\right)Q_N}{NQ_{N-1}-(N+1)Q_{N+1}+\left(\dfrac{2E}{KK'^2}-1-\dfrac{4\epsilon kH}{Fk'^2}+\dfrac{4i\lambda k}{rk'^2}\right)Q_N}. \tag{55}$$

7. The lift and moment

The lift L, and moment M, about the mid-chord point are given by

$$L = -\int_{\gamma=-2K}^{2K} p\,dx(\gamma), \qquad M = \int_{\gamma=-2K}^{2K} xp\,dx(\gamma).$$

Using equation (28) the lift may be written

$$L = -\frac{2hk}{\pi}\int_{-2K}^{2K} p\,\mathrm{sn}\,\gamma\,d\gamma. \tag{56}$$

By using the expansions (44) and (45) in equation (37) we obtain an approximate expression for the pressure p. Substituting this expression into (56) we get eventually

$$L = \frac{4Khk\rho U}{\pi^2}\int_{-2K}^{2K} (U\theta^*+\dot{G}^*)[\mathrm{cn}\,\gamma^*\,\mathrm{dn}\,\gamma^*+\mathrm{sn}\,\gamma^* Z(\gamma^*)-\epsilon^2k^2\,\mathrm{cd}\,\gamma^*]\,d\gamma^*, \tag{57}$$

which is the result correct to order k^2. Similarly the moment can, by (28), be written

$$M = 2k\left(\frac{h}{\pi}\right)^2 \int_{-2K}^{2K} p\,\mathrm{sn}\,\gamma\ln\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)d\gamma \tag{58}$$

which gives, after substituting for p,

$$M = \frac{4Kh^2k^2\rho U}{\pi^3}\int_{-2K}^{2K} (U\theta^*+\dot{G}^*)[1-2\,\mathrm{sn}^2\gamma^*+k^2(\tfrac{1}{4}-\tfrac{1}{3}\,\mathrm{sn}^4\gamma^*)]\,d\gamma^*. \tag{59}$$

For harmonic oscillations we may substitute for θ^* and $\dot{G}^*$ from (41). We find

$$L = \frac{4Khk\rho U^2}{\pi}\Bigg\{\delta(1-\epsilon^2k^2)+$$

$$+\frac{e^{i\nu t}}{\pi U}\int_{-2K}^{2K} (v_0+\dot{g}_0)[\mathrm{cn}\,\gamma^*\,\mathrm{dn}\,\gamma^*+\mathrm{sn}\,\gamma^* Z(\gamma^*)-\epsilon^2k^2\,\mathrm{cd}\,\gamma^*]\,d\gamma^*\Bigg\}, \tag{60}$$

and

$$M = \frac{4Kh^2k^2\rho U^2}{\pi^2}\left\{\delta(1+\tfrac{1}{4}k^2)+\right.$$

$$\left. +\frac{e^{i\nu t}}{\pi U}\int\limits_{-2K}^{2K}(v_0+\dot{g}_0)[1-2\,\mathrm{sn}^2\gamma^*+k^2(\tfrac{1}{4}-\tfrac{1}{3}\,\mathrm{sn}^4\gamma^*)]\,d\gamma^*\right\}, \tag{61}$$

where now
$$\dot{g}_0 \equiv \frac{2i\nu hk}{\pi U}\int\limits_0^{\gamma^*} v_0\,\mathrm{sn}\,\gamma\,d\gamma. \tag{62}$$

8. Rigid-body oscillations

As a particular case we consider a rigid aerofoil undergoing a harmonic oscillation compounded of a flapping motion parallel to the x-axis and a pitching about the mid-chord point. That is,

$$y = y^0e^{i\nu t}, \qquad \alpha = \alpha^0e^{i\nu t}$$

respectively, where y^0, α^0 are the amplitudes. The upwash velocity in this case is obtained as follows: The velocity normal to the surface of a point P on the surface distant x from the mid-point is, to first order,

$$\dot{y}-x\dot{\alpha}.$$

Again, the fluid velocity normal to the surface at this point is, to the same order,

$$U\alpha+v.$$

Equating these expressions,

$$v = -U\alpha^0e^{i\nu t}+i\nu(y^0-x\alpha^0)e^{i\nu t}.$$

That is,

$$v_0 = i\nu(y^0-x\alpha^0)-U\alpha^0 = i\nu\left[y^0-\frac{h\alpha^0}{\pi}\ln\!\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)\right]-\alpha^0U. \tag{63}$$

Substituting from (63) into (50), (51), and (62) gives

$$a_0 = (i\nu y^0-\alpha^0U)\frac{4K}{\pi U}(1-\epsilon^2k^2), \qquad b_0 = \frac{i\nu\alpha^0hk}{\pi}\frac{4K}{\pi U}(1+\tfrac{1}{4}k^2)$$

and
$$\dot{g}_0 = \frac{i\nu h}{\pi U}(i\nu y^0-\alpha^0U)\ln\!\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)+\frac{\nu^2h^2\alpha^0}{2\pi^2U}\left[\ln\!\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)\right]^2.$$

We also have the expansion (8)

$$K = \tfrac{1}{2}\pi[1+\tfrac{1}{4}k^2+O(k^4)]$$

and
$$h = \frac{\pi a}{k}[1-\tfrac{1}{3}k^2+O(k^4)]$$

which is derived from equation (7). Using these results in (60) and (61)

we eventually obtain

$$L = -\pi a\rho U e^{i\nu t}\Big\{2(i\nu y^0-\alpha^0 U)(1+T)(1+\tfrac{1}{6}k^2-2\epsilon^2k^2)+$$
$$+2i\nu a\alpha^0(1-T)(1+\tfrac{1}{12}k^2-\epsilon^2k^2)+\frac{4i\nu a}{U}(i\nu y^0-2\alpha^0 U)(1+\tfrac{1}{12}k^2-\epsilon^2k^2)\Big\} \quad (64)$$

and

$$M = -\pi a^2\rho U^2 e^{i\nu t}\Big\{\frac{2}{U}(i\nu y^0-\alpha^0 U)(1+T)(1+\tfrac{1}{12}k^2-\epsilon^2k^2)+$$
$$+\frac{2i\nu a\alpha^0}{U}(1-T)-\frac{2\nu^2a^2\alpha^0}{U^2}\Big\}. \quad (65)$$

We express the lift and moment more conveniently as non-dimensional air-load coefficients. The flapping motion is described by the coefficients l_1, l_2, m_1, m_2 and the pitching by l_3, l_4, m_3, m_4, defined by

$$\frac{L}{2\pi a\rho U^2 e^{i\nu t}} = (l_1+il_2)\frac{y^0}{2a}+(l_3+il_4)\alpha^0,$$
$$\frac{M}{4\pi a^2\rho U^2 e^{i\nu t}} = (m_1+im_2)\frac{y^0}{2a}+(m_3+im_4)\alpha^0. \quad (66)$$

Replacing ν by the dimensionless frequency parameter $\lambda = 2a\nu/U$, we find from (64)–(66),

$$\left.\begin{aligned}
l_1 &= \lambda\,\mathrm{im}\,T+\lambda^2+\tfrac{1}{12}k^2\lambda(2\,\mathrm{im}\,T+\lambda)(1-12\epsilon^2)\\
l_2 &= -\lambda(1+\mathrm{re}\,T)-\tfrac{1}{6}k^2\lambda(1+\mathrm{re}\,T)(1-12\epsilon^2)\\
l_3 &= 1+\mathrm{re}\,T-\tfrac{1}{2}\lambda\,\mathrm{im}\,T+\tfrac{1}{12}k^2(1-12\epsilon^2)(2+2\,\mathrm{re}\,T-\tfrac{1}{2}\lambda\,\mathrm{im}\,T)\\
l_4 &= \mathrm{im}\,T+\tfrac{1}{2}\lambda(3+\mathrm{re}\,T)+\tfrac{1}{12}k^2(1-12\epsilon^2)[2\,\mathrm{im}\,T+\tfrac{1}{2}\lambda(3+\mathrm{re}\,T)]\\
m_1 &= \tfrac{1}{2}\lambda\,\mathrm{im}\,T+\tfrac{1}{24}k^2(1-12\epsilon^2)\lambda\,\mathrm{im}\,T\\
m_2 &= -\tfrac{1}{2}\lambda(1+\mathrm{re}\,T)-\tfrac{1}{24}k^2(1-12\epsilon^2)\lambda(1+\mathrm{re}\,T)\\
m_3 &= \tfrac{1}{2}(1+\mathrm{re}\,T)-\tfrac{1}{4}\lambda\,\mathrm{im}\,T+\tfrac{1}{8}\lambda^2+\tfrac{1}{24}k^2(1-12\epsilon^2)(1+\mathrm{re}\,T)\\
m_4 &= \tfrac{1}{2}\,\mathrm{im}\,T-\tfrac{1}{4}\lambda(1-\mathrm{re}\,T)+\tfrac{1}{24}k^2(1-12\epsilon^2)\mathrm{im}\,T
\end{aligned}\right\}. \quad (67)$$

On putting $\epsilon = 0$, the equations (67) immediately reduce to the corresponding formulae for a solid wall tunnel (**5**).

APPENDIX 1

TRANSFORMATION OF THE SOLUTION $F(t)$

A function which satisfies, in the rectangle $-2K < \gamma < 2K$, $0 < \eta < K'$ the conditions described in section 3 above has been found (**7**) to be, in our notation,

$$F(t) = A+\frac{1}{4K}\int\limits_{-2K}^{2K}\Big\{\mathrm{im}\,F_0\frac{\vartheta_1'}{\vartheta_1}\Big[\frac{\pi}{4K}(\gamma^*-t)\Big|\tau_1\Big]-\mathrm{im}\,F_{K'}\frac{\vartheta_4'}{\vartheta_4}\Big[\frac{\pi}{4K}(\gamma^*-t)\Big|\tau_1\Big]\Big\}\,d\gamma^*+$$
$$+\frac{1}{8K}\int\limits_0^{K'}\mathrm{re}(F_{2K}-F_{-2K})\Big\{\frac{\vartheta_3'}{\vartheta_3}\Big[\frac{\pi}{4K}(i\eta^*-t)\Big|\tau_1\Big]-\frac{\vartheta_2'}{\vartheta_2}\Big[\frac{\pi}{4K}(i\eta^*-t)\Big|\tau_1\Big]\Big\}\,d\eta^*, \quad (68)$$

where the theta functions are as defined in (**9**), p. 463, and have parameter

$$\tau_1 = \frac{iK'}{2K}.$$

This solution can be transformed into an alternative form by using the properties of the theta functions given in (**9**), chap. 21. From Landen's transformation we have

$$\vartheta_1(u|\tau_1).\vartheta_2(u|\tau_1) = \text{const.}\times\vartheta_1(2u|\tau),$$

where $\tau = 2\tau_1 = iK'/K$. Differentiating logarithmically,

$$\frac{\vartheta_1'}{\vartheta_1}(u|\tau_1)+\frac{\vartheta_2'}{\vartheta_2}(u|\tau_1) = 2\frac{\vartheta_1'}{\vartheta_1}(2u|\tau) = 2i+2\frac{\vartheta_4'}{\vartheta_4}(2u+\tfrac{1}{2}\pi\tau|\tau)$$

$$= 2i+2\vartheta_3^2(0|\tau)Z[\vartheta_3^2(0|\tau)(2u+\tfrac{1}{2}\pi\tau)], \tag{69}$$

from the definition of the zeta function. An equation involving theta functions (**9**) (loc. cit.) is

$$\frac{\vartheta_4'}{\vartheta_4}(y)+\frac{\vartheta_4'}{\vartheta_4}(z)-\frac{\vartheta_4'}{\vartheta_4}(y+z) = \vartheta_2(0)\vartheta_3(0)\frac{\vartheta_1(y)\vartheta_1(z)\vartheta_1(y+z)}{\vartheta_4(y)\vartheta_4(z)\vartheta_4(y+z)}.$$

If we put $y = u+\frac{1}{2}\pi\tau$, $z = \frac{1}{2}\pi$ and use the properties of the theta functions, this becomes

$$\frac{\vartheta_1'}{\vartheta_1}(u|\tau_1)-\frac{\vartheta_2'}{\vartheta_2}(u|\tau_1) = \vartheta_2^2(0|\tau_1)\frac{\vartheta_3(u|\tau_1)\vartheta_4(u|\tau_1)}{\vartheta_1(u|\tau_1)\vartheta_2(u|\tau_1)} = \vartheta_2^2(0|\tau_1)\frac{\vartheta_3(0|\tau)}{\vartheta_2(0|\tau)}\operatorname{ns}[\vartheta_3^2(0|\tau)2u], \tag{70}$$

from Landen's transformation and the definition of the elliptic function. It is found that

$$\vartheta_3^2(0|\tau) = \frac{2K}{\pi}, \qquad \vartheta_2^2(0|\tau_1)\frac{\vartheta_3(0|\tau)}{\vartheta_2(0|\tau)} = \frac{4K}{\pi};$$

further, the periodicity of the zeta function gives

$$Z(2u+\tau K) = Z(2u+iK') = Z(2u)+\frac{\operatorname{cn}2u\operatorname{dn}2u}{\operatorname{sn}2u}-\frac{i\pi}{2K}.$$

Substituting these results into (69) and (70), we obtain

$$\frac{\vartheta_1'}{\vartheta_1}(u|\tau_1) = \frac{2K}{\pi}\left\{Z\left(\frac{4Ku}{\pi}\right)+\frac{\operatorname{cn}\frac{4Ku}{\pi}\operatorname{dn}\frac{4Ku}{\pi}+1}{\operatorname{sn}\frac{4Ku}{\pi}}\right\} \tag{71}$$

and

$$\frac{\vartheta_2'}{\vartheta_2}(u|\tau_1) = \frac{2K}{\pi}\left\{Z\left(\frac{4Ku}{\pi}\right)+\frac{\operatorname{cn}\frac{4Ku}{\pi}\operatorname{dn}\frac{4Ku}{\pi}-1}{\operatorname{sn}\frac{4Ku}{\pi}}\right\}. \tag{72}$$

By a similar procedure it can be shown that

$$\frac{\vartheta_4'}{\vartheta_4}(u|\tau_1) = \frac{2K}{\pi}\left\{Z\left(\frac{4Ku}{\pi}+iK'\right)+\frac{\operatorname{cn}\left(\frac{4Ku}{\pi}+iK'\right)\operatorname{dn}\left(\frac{4Ku}{\pi}+iK'\right)+1}{\operatorname{sn}\left(\frac{4Ku}{\pi}+iK'\right)}+\frac{\pi i}{2K}\right\}. \tag{73}$$

The theta functions of equation (68) may now be replaced by the expressions (71)–(73), and this leads immediately to the form (5).

APPENDIX 2

AN EXPRESSION FOR THE PRESSURE

For the sake of brevity we define

$$f_1(u,v) = \left(\frac{1-k\operatorname{cd}u}{1+k\operatorname{cd}u}\right)^{\epsilon}\left(\frac{1-k\operatorname{cd}v}{1+k\operatorname{cd}v}\right)^{\epsilon}\left\{\frac{\operatorname{sn}u(\operatorname{cn}v\operatorname{dn}v-\operatorname{cn}u\operatorname{dn}u)}{\operatorname{sn}^2u-\operatorname{sn}^2v}-Z(u)\right\},$$

$$f_2(u,v) = \left(\frac{1+k\operatorname{cd}u}{1-k\operatorname{cd}u}\right)^{\epsilon}\left(\frac{1-k\operatorname{cd}v}{1+k\operatorname{cd}v}\right)^{\epsilon}\left\{\frac{\operatorname{cn}v\operatorname{dn}v+\operatorname{cn}u\operatorname{dn}u}{\operatorname{sn}u-\operatorname{sn}v}+Z(u)+k\operatorname{sn}v\right\}$$

and

$$f_3(u,v) = \left(\frac{1+k\operatorname{cd}u}{1-k\operatorname{cd}u}\right)^{\epsilon}\left(\frac{1-k\operatorname{cd}v}{1+k\operatorname{cd}v}\right)^{\epsilon}\left\{\frac{\operatorname{cn}v\operatorname{dn}v+\operatorname{cn}u\operatorname{dn}u}{\operatorname{sn}u-\operatorname{sn}v}-Z(v)+k\operatorname{sn}u\right\}.$$

Then with some rearrangement equation (36) may be written

$$\Omega = \frac{1}{2\pi}\int\limits_{-2K}^{2K}\theta^* f_3(\gamma^*,\gamma)\,d\gamma^* + \frac{1}{2\pi}\int\limits_{-2K}^{2K}\theta^*\left(\frac{1+k\operatorname{cd}\gamma^*}{1-k\operatorname{cd}\gamma^*}\right)^{\epsilon}\left(\frac{1-k\operatorname{cd}\gamma}{1+k\operatorname{cd}\gamma}\right)^{\epsilon} Z(\gamma)\,d\gamma^* +$$

$$+\frac{1}{2\pi}\int\limits_0^{K'} X\operatorname{sn}i\eta^*\operatorname{ns}\gamma f_1(i\eta^*,\gamma)\,d\eta^* + \frac{1}{2\pi\operatorname{sn}\gamma}\int\limits_0^{K'} X\left(\frac{1-k\operatorname{cd}i\eta^*}{1+k\operatorname{cd}i\eta^*}\right)^{\epsilon}\left(\frac{1-k\operatorname{cd}\gamma}{1+k\operatorname{cd}\gamma}\right)^{\epsilon}\times$$

$$\times[\operatorname{cn}i\eta^*\operatorname{dn}i\eta^*+\operatorname{sn}i\eta^* Z(i\eta^*)-\operatorname{cn}\gamma\operatorname{dn}\gamma]\,d\eta^*. \quad (74)$$

From equation (26) it is clear that

$$-\frac{\operatorname{cn}\gamma\operatorname{dn}\gamma}{2\pi\operatorname{sn}\gamma}\int\limits_0^{K'} X\left(\frac{1-k\operatorname{cd}i\eta^*}{1+k\operatorname{cd}i\eta^*}\right)^{\epsilon}\left(\frac{1-k\operatorname{cd}\gamma}{1+k\operatorname{cd}\gamma}\right)^{\epsilon} d\eta^*$$

$$= \frac{\operatorname{cn}\gamma\operatorname{dn}\gamma}{2\pi\operatorname{sn}\gamma}\int\limits_{-2K}^{2K}\theta^*\left(\frac{1+k\operatorname{cd}\gamma^*}{1-k\operatorname{cd}\gamma^*}\right)^{\epsilon}\left(\frac{1-k\operatorname{cd}\gamma}{1+k\operatorname{cd}\gamma}\right)^{\epsilon} d\gamma^*,$$

while rearrangement of (31) enables us to write

$$\frac{1}{2\pi\operatorname{sn}\gamma}\int\limits_0^{K'} X\left(\frac{1-k\operatorname{cd}i\eta^*}{1+k\operatorname{cd}i\eta^*}\right)^{\epsilon}\left(\frac{1-k\operatorname{cd}\gamma}{1+k\operatorname{cd}\gamma}\right)^{\epsilon}[\operatorname{cn}i\eta^*\operatorname{dn}i\eta^*+\operatorname{sn}i\eta^* Z(i\eta^*)]\,d\eta^*$$

$$= -\frac{1}{2\pi F\operatorname{sn}\gamma}\left(\frac{1-k\operatorname{cd}\gamma}{1+k\operatorname{cd}\gamma}\right)^{\epsilon}\int\limits_0^{K'}\int\limits_{-2K}^{2K} X\operatorname{sn}i\eta^* f_1(i\eta^*,u)\,du\,d\eta^* -$$

$$-\frac{1}{2\pi F\operatorname{sn}\gamma}\left(\frac{1-k\operatorname{cd}\gamma}{1+k\operatorname{cd}\gamma}\right)^{\epsilon}\int\limits_{-2K}^{2K}\theta^*\left\{H\left(\frac{1+k\operatorname{cd}\gamma^*}{1-k\operatorname{cd}\gamma^*}\right)^{\epsilon}+\int\limits_{-2K}^{2K}\operatorname{sn}u\,f_3(\gamma^*,u)\,du\right\}d\gamma^*.$$

Using these results in (74) we find that

$$\operatorname{sn}\gamma\frac{\partial\Omega}{\partial t} = \frac{1}{2\pi}\int\limits_{-2K}^{2K}\theta^*\left\{\operatorname{sn}\gamma f_3(\gamma^*,\gamma)-\frac{1}{F}\left(\frac{1-k\operatorname{cd}\gamma}{1+k\operatorname{cd}\gamma}\right)^{\epsilon}\int\limits_{-2K}^{2K}\operatorname{sn}u\,f_3(\gamma^*,u)\,du\right\}d\gamma^* +$$

$$+\frac{1}{2\pi}\int\limits_{-2K}^{2K}\theta^*\left(\frac{1+k\operatorname{cd}\gamma^*}{1-k\operatorname{cd}\gamma^*}\right)^{\epsilon}\left(\frac{1-k\operatorname{cd}\gamma}{1+k\operatorname{cd}\gamma}\right)^{\epsilon}\left[\operatorname{cn}\gamma\operatorname{dn}\gamma+\operatorname{sn}\gamma Z(\gamma)-\frac{H}{F}\right]d\gamma^* +$$

$$+\frac{1}{2\pi}\int\limits_0^{K'}\frac{\partial X}{\partial t}\left\{\operatorname{sn}i\eta^* f_1(i\eta^*,\gamma)-\frac{1}{F}\left(\frac{1-k\operatorname{cd}\gamma}{1+k\operatorname{cd}\gamma}\right)^{\epsilon}\int\limits_{-2K}^{2K}\operatorname{sn}i\eta^* f_1(i\eta^*,u)\,du\right\}d\eta^*. \quad (75)$$

Now equation (33) gives that

$$\frac{\partial X}{\partial t} = -\frac{\pi U}{2hk}\, i \operatorname{ns} i\eta \frac{\partial X}{\partial \eta},$$

and hence the last integral of (75) can be written

$$I_1 = -\frac{U}{4hk}\int_0^{K'} i\frac{\partial X}{\partial \eta^*}\left\{f_1(i\eta^*,\gamma)-\frac{1}{F}\left(\frac{1-k\operatorname{cd}\gamma}{1+k\operatorname{cd}\gamma}\right)^{\epsilon}\int_{-2K}^{2K} f_1(i\eta^*,u)\,du\right\}d\eta^*$$

$$= \frac{U}{4hk}\int_0^{K'} X\left\{i\frac{\partial}{\partial \eta^*}f_1(i\eta^*,\gamma)-\frac{1}{F}\left(\frac{1-k\operatorname{cd}\gamma}{1+k\operatorname{cd}\gamma}\right)^{\epsilon}\int_{-2K}^{2K} i\frac{\partial}{\partial \eta^*}f_1(i\eta^*,u)\,du\right\}d\eta^*,$$

the integrated term vanishing. It can be verified by differentiation that

$$i\frac{\partial}{\partial \eta}f_1(i\eta,v) = -\frac{\partial}{\partial v}f_1(v,i\eta)-2\epsilon k\left(\frac{1-k\operatorname{cd} i\eta}{1+k\operatorname{cd} i\eta}\right)^{\epsilon}\left(\frac{1-k\operatorname{cd} v}{1+k\operatorname{cd} v}\right)^{\epsilon}\times$$
$$\times[\operatorname{cn} v\operatorname{dn} v+\operatorname{sn} v Z(v)-\operatorname{cn} i\eta\operatorname{dn} i\eta-\operatorname{sn} i\eta Z(i\eta)].$$

Thus we obtain

$$I_1 = -\frac{U}{4hk}\int_0^{K'} X\frac{\partial}{\partial \gamma}f_1(\gamma,i\eta^*)\,d\eta^*+\frac{\epsilon U}{2h}\int_{-2K}^{2K}\theta^*\left(\frac{1+k\operatorname{cd}\gamma^*}{1-k\operatorname{cd}\gamma^*}\right)^{\epsilon}\left(\frac{1-k\operatorname{cd}\gamma}{1+k\operatorname{cd}\gamma}\right)^{\epsilon}\times$$
$$\times\left[\operatorname{cn}\gamma\operatorname{dn}\gamma+\operatorname{sn}\gamma Z(\gamma)-\frac{H}{F}\right]d\gamma^*, \quad (76)$$

where the last term is found by replacement from (26), and other terms have cancelled out.

From the above definitions it is readily seen that

$$\operatorname{sn} v f_3(u,v) = \operatorname{sn} u f_2(u,v)-\left(\frac{1+k\operatorname{cd} u}{1-k\operatorname{cd} u}\right)^{\epsilon}\left(\frac{1-k\operatorname{cd} v}{1+k\operatorname{cd} v}\right)^{\epsilon}\times$$
$$\times[\operatorname{cn} u\operatorname{dn} u+\operatorname{sn} u Z(u)+\operatorname{cn} v\operatorname{dn} v+\operatorname{sn} v Z(v)].$$

This allows the first two terms of (75) to be written

$$I_2 = \frac{1}{2\pi}\int_{-2K}^{2K}\theta^*\left\{\operatorname{sn}\gamma^* f_2(\gamma^*,\gamma)-\frac{1}{F}\left(\frac{1-k\operatorname{cd}\gamma}{1+k\operatorname{cd}\gamma}\right)^{\epsilon}\int_{-2K}^{2K}\operatorname{sn}\gamma^* f_2(\gamma^*,u)\,du\right\}d\gamma^*.$$

Introduce now the function

$$G(\gamma^*) = \frac{2hk}{\pi}\int_0^{\gamma^*}\theta^*\operatorname{sn}\gamma^*\,d\gamma^*;$$

then we can put

$$I_2 = \frac{1}{4hk}\int_{-2K}^{2K}\frac{\partial G^*}{\partial \gamma^*}\left\{f_2(\gamma^*,\gamma)-\frac{1}{F}\left(\frac{1-k\operatorname{cd}\gamma}{1+k\operatorname{cd}\gamma}\right)^{\epsilon}\int_{-2K}^{2K} f_2(\gamma^*,u)\,du\right\}d\gamma^*$$

$$= -\frac{1}{4hk}\int_{-2K}^{2K} G^*\left\{\frac{\partial}{\partial \gamma^*}f_2(\gamma^*,\gamma)-\frac{1}{F}\left(\frac{1-k\operatorname{cd}\gamma}{1+k\operatorname{cd}\gamma}\right)^{\epsilon}\int_{-2K}^{2K}\frac{\partial}{\partial \gamma^*}f_2(\gamma^*,u)\,du\right\}d\gamma^*,$$

the integrated term vanishing. Again it can be shown that

$$\frac{\partial}{\partial \gamma^*}f_2(\gamma^*,v) = -\frac{\partial}{\partial v}f_3(\gamma^*,v)-2\epsilon k\left(\frac{1+k\operatorname{cd}\gamma^*}{1-k\operatorname{cd}\gamma^*}\right)^{\epsilon}\left(\frac{1-k\operatorname{cd} v}{1+k\operatorname{cd} v}\right)^{\epsilon}\times$$
$$\times[\operatorname{cn}\gamma^*\operatorname{dn}\gamma^*+\operatorname{sn}\gamma^* Z(\gamma^*)+\operatorname{cn} v\operatorname{dn} v+\operatorname{sn} v Z(v)].$$

Hence we find

$$I_2 = \frac{1}{4hk}\int_{-2K}^{2K} \dot{G}^*\left\{\frac{\partial}{\partial\gamma}f_3(\gamma^*,\gamma)+2\epsilon k\left(\frac{1+k\,\mathrm{cd}\,\gamma^*}{1-k\,\mathrm{cd}\,\gamma^*}\right)^\epsilon\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)^\epsilon\times\right.$$

$$\left.\times\left[\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma+\mathrm{sn}\,\gamma Z(\gamma)-\frac{H}{F}\right]\right\}d\gamma^*. \quad (77)$$

From (76) and (77) we obtain

$$\frac{2hk\rho U}{\pi}\int_0^\gamma \mathrm{sn}\,\gamma\frac{\partial\Omega}{\partial t}\,\partial\gamma = -\frac{\rho U^2}{2\pi}\int_0^{K'} Xf_1(\gamma,i\eta^*)\,d\eta^*+\frac{\rho U}{2\pi}\int_{-2K}^{2K}\dot{G}^*f_3(\gamma^*,\gamma)\,d\gamma^*+$$

$$+\frac{\epsilon k\rho U}{\pi}\int_{-2K}^{2K}(U\theta^*+\dot{G}^*)\left(\frac{1+k\,\mathrm{cd}\,\gamma^*}{1-k\,\mathrm{cd}\,\gamma^*}\right)^\epsilon\int_0^\gamma\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)^\epsilon\left[\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma+\mathrm{sn}\,\gamma Z(\gamma)-\frac{H}{F}\right]d\gamma d\gamma^*.$$

Adding this to $\rho U^2\Omega$, given from (36), yields the required formula for the pressure.

APPENDIX 3

EVALUATION OF AN INTEGRAL

Let

$$J = \int_0^{K'}\left(\frac{1-k\,\mathrm{cd}\,i\eta}{1+k\,\mathrm{cd}\,i\eta}\right)^{\epsilon+\alpha}\int_{-2K}^{2K}\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)^\epsilon\left\{\mathrm{sn}^2\gamma\frac{\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma-\mathrm{cn}\,i\eta\,\mathrm{dn}\,i\eta}{\mathrm{sn}^2 i\eta-\mathrm{sn}^2\gamma}+\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma\right\}d\gamma d\eta,$$

where $\alpha = i\nu h/\pi U$. This may be written

$$J = I+H\int_0^{K'}\left(\frac{1-k\,\mathrm{cd}\,i\eta}{1+k\,\mathrm{cd}\,i\eta}\right)^{\epsilon+\alpha}d\eta,$$

where

$$I = \int_0^{K'}\left(\frac{1-k\,\mathrm{cd}\,i\eta}{1+k\,\mathrm{cd}\,i\eta}\right)^{\epsilon+\alpha}\int_{-2K}^{2K}\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)^\epsilon \mathrm{sn}\,\gamma\, g(\gamma,i\eta)\,d\gamma d\eta$$

and

$$g(u,v) \equiv \mathrm{sn}\,u\,\frac{\mathrm{cn}\,v\,\mathrm{dn}\,v-\mathrm{cn}\,u\,\mathrm{dn}\,u}{\mathrm{sn}^2u-\mathrm{sn}^2v}-Z(u).$$

Now

$$I = \frac{1}{2k\epsilon}\int_0^{K'}\int_{-2K}^{2K}\left(\frac{1-k\,\mathrm{cd}\,i\eta}{1+k\,\mathrm{cd}\,i\eta}\right)^{\epsilon+\alpha}g(\gamma,i\eta)\frac{\partial}{\partial\gamma}\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)^\epsilon d\gamma d\eta$$

$$= -\frac{1}{2k\epsilon}\int_0^{K'}\int_{-2K}^{2K}\left(\frac{1-k\,\mathrm{cd}\,i\eta}{1+k\,\mathrm{cd}\,i\eta}\right)^{\epsilon+\alpha}\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)^\epsilon\frac{\partial}{\partial\gamma}g(\gamma,i\eta)\,d\gamma d\eta,$$

the integrated term vanishing. We can show by differentiation that

$$\frac{\partial}{\partial\gamma}g(\gamma,i\eta) = -i\frac{\partial}{\partial\eta}g(i\eta,\gamma).$$

Hence

$$I = \frac{i}{2k\epsilon}\int_0^{K'}\int_{-2K}^{2K}\left(\frac{1-k\,\mathrm{cd}\,i\eta}{1+k\,\mathrm{cd}\,i\eta}\right)^{\epsilon+\alpha}\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)^\epsilon\frac{\partial}{\partial\eta}g(i\eta,\gamma)\,d\gamma d\eta$$

$$= \frac{\epsilon+\alpha}{\epsilon}\int_0^{K'}\int_{-2K}^{2K}\left(\frac{1-k\,\mathrm{cd}\,i\eta}{1+k\,\mathrm{cd}\,i\eta}\right)^{\epsilon+\alpha}\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)^\epsilon \mathrm{sn}\,i\eta\, g(i\eta,\gamma)\,d\gamma d\eta.$$

From the definition of g we have

$$\operatorname{sn} i\eta\, g(i\eta,\gamma) = \operatorname{sn}\gamma\, g(\gamma, i\eta)+\operatorname{cn}\gamma\operatorname{dn}\gamma+\operatorname{sn}\gamma Z(\gamma)-\operatorname{cn} i\eta\operatorname{dn} i\eta-\operatorname{sn} i\eta Z(i\eta),$$

so that

$$I = \frac{\epsilon+\alpha}{\epsilon}\left[I+H\int\limits_0^{K'}\left(\frac{1-k\operatorname{cd} i\eta}{1+k\operatorname{cd} i\eta}\right)^{\epsilon+\alpha}d\eta-\right.$$
$$\left.-F\int\limits_0^{K'}\left(\frac{1-k\operatorname{cd} i\eta}{1+k\operatorname{cd} i\eta}\right)^{\epsilon+\alpha}\{\operatorname{cn} i\eta\operatorname{dn} i\eta+\operatorname{sn} i\eta Z(i\eta)\}\,d\eta\right].$$

This gives

$$I = \frac{\epsilon+\alpha}{\alpha}\left\{F\int\limits_0^{K'}\left(\frac{1-k\operatorname{cd} i\eta}{1+k\operatorname{cd} i\eta}\right)^{\epsilon+\alpha}[\operatorname{cn} i\eta\operatorname{dn} i\eta+\operatorname{sn} i\eta Z(i\eta)]\,d\eta-H\int\limits_0^{K'}\left(\frac{1-k\operatorname{cd} i\eta}{1+k\operatorname{cd} i\eta}\right)^{\epsilon+\alpha}d\eta\right\},$$

which leads directly to the required expression for J.

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THE TWO-DIMENSIONAL LAMINAR FLOW NEAR THE STAGNATION POINT OF A CYLINDER WHICH HAS AN ARBITRARY TRANSVERSE MOTION

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SUMMARY

Exact solutions of the Navier–Stokes equations are derived for two-dimensional flow against an infinite flat plate normal to the stream and performing an arbitrary transverse motion. This generalizes Glauert's solution for an oscillating transverse motion.

After a brief approximate investigation of the reaction of an arbitrary boundary layer to arbitrary changes in the free-stream velocity, the problem described above is considered. When the wall moves from rest, then, by means of a Laplace-transform technique, expansions of the velocity distribution for small and large times are given in terms of the velocity of the wall. Further, it is indicated how a Pohlhausen type of method may always be used to obtain an approximate solution for the purpose of linking up these expansions across the range of times for which neither is valid. The method may also be used for motions other than those starting from rest. In uniform motion started impulsively from rest, the expansions overlap and the approximate solution agrees well with them.

1. Introduction

For many years the study of unsteady laminar boundary layers was restricted to problems of boundary-layer growth in motion from rest and the boundary layer in oscillatory motions with zero mean velocity. Recently Lighthill (**1**), Stuart (**2**), Glauert (**3**), and Rott (**4**) have investigated how the boundary layer on a two-dimensional body reacts to fluctuations in the main stream. Glauert (**3**) and Rott (**4**) in particular studied the two-dimensional flow when an infinite plane wall normal to the main stream makes transverse oscillations in its own plane.

Watson (**5**) has generalized the solution given by Stuart (**2**) to the case of a main-stream flow which varies arbitrarily with time. In the case of an impulsive increase of the main-stream velocity from one value to another, it was found that the boundary layer initially reacts as though it were inviscid, except within a narrow region adjacent to the surface. Here a secondary boundary layer of the classic Rayleigh impulsive type is formed. As time progresses the secondary layer increases in thickness and gradually distorts the flow until the ultimate steady-state boundary layer is attained.

The primary purpose of this paper is to generalize the work of Glauert

(**3**) and Rott (**4**) to the case of arbitrary motions of the wall. However, a preliminary general analysis of the unsteady two-dimensional laminar boundary-layer equations is given in section 2. For an arbitrary motion of a body commencing at $t = 0$, the first term in a series expansion for the velocity in the boundary layer at small times is obtained by omitting the convection terms. Further, for the particular case of a small impulsive increase in free-stream velocity the first term in an expansion for large times is derived from the quasi-steady solution given by Lighthill.

The generalization of Glauert's problem is considered in section 3, where a linear partial differential equation is obtained. In section 4, by using the Laplace-transform technique, the velocity, as a series expansion for small times and as an asymptotic expansion for large times, is easily derived from Glauert's solutions for oscillations of small and large frequencies. Numerical results are given for the case of uniform motion of the wall, started impulsively from rest. Examination of the curve of skin friction against time shows that the expansion for small times tends smoothly to the constant limiting value for large times.

In general, however, we cannot expect the two solutions to join quite so conveniently, for which reason an alternative approximate solution valid at all times is given in section 5. This is a Pohlhausen type of method in which the differential equation describing the unsteady flow is satisfied only at the wall, at infinity, and on an average by integrating it across the boundary layer. This approximate solution, being valid at all times, may be used to interpolate accurately between the two expansions whenever there is an interval of time for which neither is applicable.

For impulsive motion of the wall the curves of skin friction given by the exact (Laplace) and approximate (Pohlhausen) methods are near to each other and, to a good approximation, are parallel. This encourages us to hope that in other cases reasonably easy interpolation will be possible, particularly as a similar result was obtained by Glauert and Lighthill (**6**) in using a Pohlhausen method to join two expansions for the axisymmetric boundary layer on a long circular cylinder. For impulsive motion of the wall the characteristic time t_c representative of the time taken for the boundary layer to settle down to its steady limiting state is of order δ_{10}^2/ν, where δ_{10} is the undisturbed displacement thickness. A similar result was found for the asymptotic suction profile (Watson (**5**)).

2. An analysis of the unsteady two-dimensional laminar boundary layer equations

Let x denote the distance along a body, z the distance normal to it, u and w the corresponding components of velocity, t the time, p the

pressure, ρ the density, and ν the kinematic viscosity. For incompressible flow, the equations of the boundary layer and of continuity are

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}+\nu\frac{\partial^2 u}{\partial z^2}, \tag{2.1}$$

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0, \tag{2.2}$$

subject to the boundary conditions

$$u=w=0 \quad \text{at} \quad z=0, \qquad u\to U(x,t) \quad \text{as} \quad z\to\infty, \tag{2.3}$$

where $U(x,t)$ denotes the velocity at the edge of the boundary layer.

We now write

$$\left.\begin{aligned} u(x,z,t) &= u_0(x,z)+u_1(x,z,t)\\ w(x,z,t) &= w_0(x,z)+w_1(x,z,t)\\ U(x,t) &= U_0(x)+U_1(x,t) \end{aligned}\right\}, \tag{2.4}$$

where the suffix 0 denotes a basic or original steady motion, and the suffix 1 denotes the time-dependent part of the motion. The steady part of the motion satisfies the equations

$$\left.\begin{aligned} u_0\frac{\partial u_0}{\partial x}+w_0\frac{\partial u_0}{\partial z} &= U_0\frac{dU_0}{dx}+\nu\frac{\partial^2 u_0}{\partial z^2}\\ \frac{\partial u_0}{\partial x}+\frac{\partial w_0}{\partial z} &= 0 \end{aligned}\right\} \tag{2.5}$$

subject to

$$u_0=w_0=0 \quad \text{at} \quad z=0, \qquad u_0\to U_0(x) \quad \text{as} \quad z\to\infty. \tag{2.6}$$

When equations (2.4) are substituted into (2.1) to (2.3), and (2.5) and (2.6) are used, we find that

$$\begin{aligned} \frac{\partial u_1}{\partial t}+u_0\frac{\partial u_1}{\partial x}+u_1\frac{\partial u_0}{\partial x}+w_0\frac{\partial u_1}{\partial z}+w_1\frac{\partial u_0}{\partial z}-\frac{\partial U_1}{\partial t}-U_0\frac{\partial U_1}{\partial x}-U_1\frac{dU_0}{dx}-\nu\frac{\partial^2 u_1}{\partial z^2}\\ = U_1\frac{\partial U_1}{\partial x}-u_1\frac{\partial u_1}{\partial x}-w_1\frac{\partial u_1}{\partial z}, \end{aligned} \tag{2.7}$$

$$\frac{\partial u_1}{\partial x}+\frac{\partial w_1}{\partial z}=0, \tag{2.8}$$

subject to the boundary conditions

$$u_1=w_1=0 \quad \text{at } z=0, \qquad u_1\to U_1(x,t) \quad \text{as } z\to\infty. \tag{2.9}$$

We now suppose that the flow is steady for $t\leqslant 0$, so that the time-dependent motion commences at $t=0$. Additional vorticity will be created at the surface of the body and will diffuse outwards, reacting with the main boundary layer. This region of additional vorticity created by

the time-dependent part of the motion will be termed the 'secondary boundary layer', outside of which the flow reacts to the time-dependent part of the motion as if inviscid; accordingly, in this region $u_1 = U_1(x,t)$. For t small the secondary layer will be very thin and the viscous term in (2.7) very large compared with the convection terms. Thus for such values of t equation (2.7) may be replaced by

$$\frac{\partial u_1}{\partial t}-\frac{\partial U_1}{\partial t} = \nu\frac{\partial^2 u_1}{\partial z^2} \quad (z>0,\ t>0),$$

which may be written in the form

$$\frac{\partial(U_1-u_1)}{\partial t} = \nu\frac{\partial^2(U_1-u_1)}{\partial z^2} \quad (z>0,\ t>0) \tag{2.10}$$

subject to

$$\begin{aligned} &U_1-u_1 = U_1(x,t) && \text{when } z=0,\ t>0,\\ &U_1-u_1 \to 0 && \text{as } \quad z\to\infty,\ t>0,\\ &U_1-u_1 = 0 && \text{when } z>0,\ t=0. \end{aligned}$$

The solution of (2.10) (cf. (7) p. 334) which satisfies these boundary conditions is

$$u_1 = U_1(x,t)-\frac{z}{2(\pi\nu)^{\frac{1}{2}}}\int\limits_0^t U_1(x,t-\tau)\frac{e^{-z^2/4\nu\tau}}{\tau^{\frac{3}{2}}}\,d\tau. \tag{2.11}$$

From (2.8), (2.9), and (2.11) it can be shown that

$$w_1 = -z\frac{\partial U_1(x,t)}{\partial x}+\left(\frac{\nu}{\pi}\right)^{\frac{1}{2}}\int\limits_0^t \frac{\partial U_1(x,t-\tau)}{\partial x}\,\frac{(1-e^{-z^2/4\nu\tau})}{\tau^{\frac{1}{2}}}\,d\tau. \tag{2.12}$$

Provided that t is small enough for equation (2.10) to be valid, the total velocity components are given by (2.4), (2.11), and (2.12).

It follows from (2.4) that the skin friction is given by

$$\tau_w = \tau_{w_0}+\mu\left(\frac{\partial u_1}{\partial z}\right)_{z=0}, \tag{2.13}$$

where τ_{w_0} is the steady skin-friction. To evaluate (2.13) for the particular solution (2.11), it is sufficient to assume the mild condition

$$|U_1(x,t)-U_1(x,t-\tau)| \leqslant K|\tau|^k,$$

where K, k are positive constants and $k>\frac{1}{2}$; then it can readily be shown that the skin friction, for small and positive times, is given by

$$(\tau_w-\tau_{w_0})/\mu = \frac{U_1(x,t)}{(\pi\nu t)^{\frac{1}{2}}}+\frac{1}{2(\pi\nu)^{\frac{1}{2}}}\int\limits_0^t \{U_1(x,t)-U_1(x,t-\tau)\}\frac{d\tau}{\tau^{\frac{3}{2}}}. \tag{2.14}$$

Consider now the particular case in which the free-stream velocity increases impulsively by $\Delta U_0(x)$ at $t = 0$ and remains constant so that

$$U_1(x,t) = 0 \qquad \text{for } t \leqslant 0,$$
$$U_1(x,t) = \Delta U_0(x) \quad \text{for } t > 0.$$

In this case (2.11) yields

$$u_1 = \Delta U_0(x)\operatorname{erf}\{z/2(\nu t)^{\frac{1}{2}}\},$$

where
$$\operatorname{erf} x = 2\pi^{-\frac{1}{2}}\int\limits_0^x e^{-x^2}\,dx.$$

Similarly, by using (2.12) or by using (2.8) directly, we can obtain a formula for the velocity component w_1. The skin friction is determined immediately from (2.14) as $(\tau_w - \tau_{w_0})/\mu = \Delta U_0(x)(\pi\nu t)^{-\frac{1}{2}}$ for small times. This secondary boundary layer is, of course, the Rayleigh impulsive flow; the theory shows that, initially, it may be added linearly to the original steady flow.

For large values of the time the velocity distribution will tend to that for steady flow with free-stream velocity

$$U(x,t) = (1+\Delta)U_0(x).$$

If Δ is small compared with unity then u_1, w_1 will be small compared with u_0, w_0. On neglecting the squares and products of u_1, w_1, U_1 in (2.7) we deduce that

$$u_1 = \Delta\left(u_0 + \tfrac{1}{2}z\frac{\partial u_0}{\partial z}\right), \qquad w_1 = \tfrac{1}{2}\Delta\left(w_0 + z\frac{\partial w_0}{\partial z}\right), \tag{2.15}$$

the 'quasi-steady' solution obtained by Lighthill (1). The skin friction is found to be $\tau_w = \tau_{w_0} + 3\Delta\tau_{w_0}/2$.

The complete solution of equations (2.7) and (2.8) for all times presents considerable difficulties even in the case of impulsive flow. A simple particular case, that of flow near a stagnation point when the wall is impulsively set into motion with a steady velocity, is solved in the following sections; this is equivalent to the case of $U_0(x) \propto x$ with $U_1 = 0$ for $t \leqslant 0$ and $U_1 =$ constant for $t > 0$. The method is applicable to other cases of unsteady motion of the wall corresponding to the same $U_0(x)$, but with $U_1(x,t)$ as a function of time only. Because equation (2.7) is linear in these cases, exact solutions can be obtained; furthermore the solutions also satisfy the Navier–Stokes equations exactly.

3. A general analysis of the flow near a stagnation point when the wall is in unsteady motion in its own plane

The solution for the steady boundary layer near a stagnation point on an infinite plate normal to the stream has long been known, and recently

Glauert (3) and Rott (4) have considered the unsteady flow which arises when the wall oscillates in its own plane. The solution, like that for the steady flow, is an exact solution of the Navier–Stokes equations. We will consider here a generalization of this oscillatory solution to the case in which the wall moves in an arbitrary manner.

With coordinates and velocity components defined as in section 2, a steady solution of the Navier–Stokes and continuity equations is known in the form

$$U_0 = ax, \qquad u_0 = ax\phi'(\eta), \qquad w_0 = -(a\nu)^{\frac{1}{2}}\phi(\eta), \qquad \eta = z(a/\nu)^{\frac{1}{2}}, \quad (3.1)$$

where a is a constant and ϕ satisfies

$$\left.\begin{aligned} \phi'''+\phi\phi''+1-\phi'^2 &= 0 \\ \phi(0) = \phi'(0) = 0; \qquad \phi'(\infty) &= 1 \end{aligned}\right\}. \quad (3.2)$$

subject to the conditions

The corresponding pressure distribution (p) can also be calculated and is given by

$$p-p_0 = -\tfrac{1}{2}\rho[a^2x^2+a\nu(\phi^2+2\phi')], \quad (3.3)$$

where ρ is the density and p_0 the stagnation pressure.

For the case of flow in which the wall has a velocity which is an arbitrary function of time, it is possible to find an exact solution of the Navier–Stokes and continuity equations relative to axes fixed in space in the form

$$u = ax\phi'(\eta)+\epsilon F(\eta,\tau), \qquad w = -(a\nu)^{\frac{1}{2}}\phi(\eta), \qquad \tau = at, \quad (3.4)$$

where ϵ is a reference velocity for the wall and F is given by the equation

$$F''+\phi F'-\phi' F-\frac{\partial F}{\partial \tau} = 0. \quad (3.5)$$

Primes denote differentiation with respect to η and the boundary conditions are

$$F = F_w(\tau) \quad \text{at } \eta = 0; \qquad F \to 0 \quad \text{as } \eta \to \infty, \quad (3.6)$$

where $\epsilon F_w(\tau)$ is the velocity of the wall. The pressure distribution is still given by equation (3.3). When (3.5), (3.6) are solved in conjunction with (3.2), the solution for arbitrary motion of the wall is obtained.

Glauert (3) has investigated the case when the wall oscillates harmonically in which case (3.5) is separable, leaving an ordinary differential equation to be integrated with η as the independent variable. This same differential equation of course arises when the Fourier or Laplace transform is applied to (3.5). The latter will be used in this paper.

4. Solution by Laplace-transform technique

We suppose that the motion of the wall starts at t (or τ) $= 0$. The Laplace transform, $\bar{x}(p)$, of the function $x(\tau)$ is defined (7) to be

$$\bar{x}(p) = \int\limits_0^\infty e^{-p\tau}x(\tau)\,d\tau, \qquad \mathrm{re}(p) > 0. \tag{4.1}$$

The inverse of this is

$$x(\tau) = \frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda\tau}\bar{x}(\lambda)\,d\lambda \quad (\gamma > c > 0). \tag{4.2}$$

On multiplying (3.5) by $e^{-p\tau}$ and integrating with respect to τ from 0 to ∞ we obtain

$$\bar{F}''+\phi\bar{F}'-\phi'\bar{F}-p\bar{F} = 0, \tag{4.3}$$

assuming that $e^{-p\tau}F(\eta,\tau) \to 0$ as $\tau \to \infty$. From (3.6) the boundary conditions on $\bar{F}(\eta,p)$ are

$$\bar{F} = \bar{F}_w(p) \quad \text{at } \eta = 0; \qquad \bar{F} \to 0 \quad \text{as } \eta \to \infty. \tag{4.4}$$

Accordingly the function

$$\bar{f}(\eta,p) = \bar{F}(\eta,p)/\bar{F}_w(p) \tag{4.5}$$

satisfies

$$\bar{f}''+\phi\bar{f}'-\phi'\bar{f}-p\bar{f} = 0, \tag{4.6}$$

subject to the boundary conditions

$$\left.\begin{aligned}\bar{f} &= 1 \quad \text{at } \eta = 0\\ \bar{f} &\to 0 \quad \text{as } \eta \to \infty\end{aligned}\right\}. \tag{4.7}$$

A general solution of (4.6), (4.7) has not been obtained. However, an expansion of $\bar{f}$ for small and large p yields a formal expansion of F for small and large τ. The solution for small τ will be obtained in section 4.1, and that for large τ in section 4.2.

4.1. *Solution for small times*

If $\bar{f}(\eta,p)$ is expanded in a series for large p, and the resulting series is inverted term by term, then a formal solution of (3.5), (3.6) will be obtained for small positive τ (Carslaw and Jaeger (7)). Now Glauert (3) has considered (4.6), (4.7) with p replaced by $(i\omega/c)$. From his results we deduce that a solution of (4.6), (4.7) for large values of p can be obtained in the form

$$\bar{f}(\eta,p) = \sum_{n=0}^{\infty} p^{-\frac{1}{2}n}e^{-p^{\frac{1}{2}}\eta}\sum_{m=0}^{n} a_{m,n}\,p^{\frac{1}{2}m}\eta^m$$

which can be rewritten

$$\bar{f}(\eta,p) = \Big(\sum_{m=0}^{\infty} a_{m,m}\,\eta^m\Big)e^{-p^{\frac{1}{2}}\eta}+\Big(\sum_{m=0}^{\infty} a_{m,m+1}\,\eta^m\Big)p^{-\frac{1}{2}}e^{-p^{\frac{1}{2}}\eta}+$$
$$+\sum_{n=0}^{\infty}\Big(\sum_{m=0}^{\infty} a_{m,m+n+2}\,\eta^m\Big)p^{-1-\frac{1}{2}n}e^{-p^{\frac{1}{2}}\eta} \tag{4.8}$$

with a change in the definition of n. The coefficients $a_{m,n}$ are given in Table 1. We note that the inverses of $e^{-p^{\frac{1}{2}}\eta}$, $p^{-\frac{1}{2}}e^{-p^{\frac{1}{2}}\eta}$, $p^{-1-\frac{1}{2}n}e^{-p^{\frac{1}{2}}\eta}$ $(n = 0, 1, 2,...)$ are [cf. (**7**), Appendix III] $\frac{1}{2}\pi^{-\frac{1}{2}}\tau^{-\frac{3}{2}}\eta e^{-\eta^2/4\tau}$, $(\pi\tau)^{-\frac{1}{2}}e^{-\eta^2/4\tau}$, $(4\tau)^{\frac{1}{2}n}\mathrm{i}^n\mathrm{erfc}(\eta/2\tau^{\frac{1}{2}})$ respectively, where

$$\mathrm{i}^0\mathrm{erfc}\, x = \mathrm{erfc}\, x$$

and

$$\mathrm{i}^n\mathrm{erfc}\, x = \left\{\int_x^\infty\right\}^n \mathrm{erfc}\, x\, dx.$$

TABLE 1

Values of $a_{m,m+r}$ from ref. (**3**)

$m \backslash r$	0	1	2	3	4	5	6	7	8
0	1	0	0	0	0	0	0	0	0
1	0	0	$-\frac{3}{8}A$	$\frac{3}{16}$	0	$\frac{33}{128}A^2$	$-\frac{65}{128}A$	$\frac{139}{512}$	..
2	0	$-\frac{3}{8}A$	$\frac{3}{16}$	0	$\frac{33}{128}A^2$	$-\frac{65}{128}A$	$\frac{139}{512}$	..	..
3	$-\frac{1}{12}A$	$\frac{1}{8}$	0	$\frac{11}{64}A^2$	$-\frac{65}{192}A$	$\frac{139}{768}$	..	..	..
4	$\frac{1}{48}$	0	$\frac{9}{128}A^2$	$-\frac{31}{192}A$	$\frac{139}{1536}$	..	..	..	..
5	0	$\frac{3}{160}A^2$	$-\frac{103}{1920}A$	$\frac{127}{3840}$	..	..	..	..	..
6	$\frac{1}{360}A^2$	$-\frac{73}{5760}A$	$\frac{103}{11520}$	..	..	..	..	..	..
7	$-\frac{31}{20160}A$	$\frac{73}{40320}$	..	..	..	..	..	..	..
8	$\frac{31}{161280}$	..	..	..	..	..	..	..	..

In this Table the constant A is given by $A = \phi''(0) = 1{\cdot}2326$.

Minor numerical errors in some of the coefficients from equations (25) to (33) of ref. **3** are corrected: Mr. Glauert has kindly agreed with these corrections.

If $f(\eta, \tau)$ denotes the inverse of $\bar{f}(\eta, p)$, then by formally inverting (4.8) term by term we obtain

$$f(\eta,\tau) = \tfrac{1}{2}\pi^{-\frac{1}{2}}\Big(\sum_{m=0}^{\infty} a_{m,m}\,\eta^{m+1}\Big)\tau^{-\frac{3}{2}}e^{-\eta^2/4\tau} + \pi^{-\frac{1}{2}}\Big(\sum_{m=0}^{\infty} a_{m,m+1}\,\eta^{m}\Big)\tau^{-\frac{1}{2}}e^{-\eta^2/4\tau} +$$

$$+\sum_{n=0}^{\infty} 2^n\Big(\sum_{m=0}^{\infty} a_{m,m+n+2}\,\eta^m\Big)\tau^{\frac{1}{2}n}\mathrm{i}^n\mathrm{erfc}(\eta/2\tau^{\frac{1}{2}})$$

$$= A_1(\eta)\tau^{-\frac{3}{2}}e^{-\eta^2/4\tau} + A_2(\eta)\tau^{-\frac{1}{2}}e^{-\eta^2/4\tau} + \sum_{n=0}^{\infty} B_n(\eta)\tau^{\frac{1}{2}n}\mathrm{i}^n\mathrm{erfc}\,\frac{\eta}{2\tau^{\frac{1}{2}}} \tag{4.9}$$

say. On inverting (4.5) we obtain ((**7**), Theorem VI)

$$F(\eta,\tau) = \int_0^\tau F_w(\tau - s)f(\eta, s)\,ds. \tag{4.10}$$

The skin friction is given by (3.4) as

$$\tau_w = \tau_{w_0} + \epsilon\mu\left(\frac{a}{\nu}\right)^{\frac{1}{2}}\left(\frac{\partial F}{\partial \eta}\right)_{\eta=0}, \tag{4.11}$$

where $\tau_{w_0} = \mu a^{\frac{3}{2}}\nu^{-\frac{1}{2}}x\phi''(0)$, and from (4.10) with $(\partial F/\partial\eta)_{\eta=0}$ written as a limit we have

$$\left(\frac{\partial F}{\partial \eta}\right)_{\eta=0} = F_w(\tau)\left[-\frac{1}{(\pi\tau)^{\frac{1}{2}}} + 2a_{1,2}\left(\frac{\tau}{\pi}\right)^{\frac{1}{2}} + \sum_{n=0}^{\infty} \frac{a_{1,n+3}\,\tau^{1+\frac{1}{2}n}}{\Gamma(2+\frac{1}{2}n)}\right] -$$

$$-\int_0^{\tau} \{F_w(\tau)-F_w(\tau-s)\}\left\{\frac{1}{2\pi^{\frac{1}{2}}s^{\frac{3}{2}}} + \frac{a_{1,2}}{(\pi s)^{\frac{1}{2}}} + \sum_{n=0}^{\infty}\frac{a_{1,n+3}\,s^{\frac{1}{2}n}}{\Gamma(1+\frac{1}{2}n)}\right\}ds, \tag{4.12}$$

in which the coefficients $a_{m,n}$ are given in Table 1.

Consider now the particular problem of a wall at rest, impulsively set in uniform motion at $\tau = 0$ with velocity ϵ, that is, $F_w(\tau)$ equals the unit function $H(\tau)$. Then the velocity distribution becomes, by (4.10), (4.9),

$$F(\eta,\tau) = \int_0^{\tau} f(\eta,s)\,ds$$

$$= A_1(\eta)\int_0^{\tau} s^{-\frac{3}{2}}e^{-\eta^2/4s}\,ds + A_2(\eta)\int_0^{\tau} s^{-\frac{1}{2}}e^{-\eta^2/4s}\,ds +$$

$$+\sum_{n=0}^{\infty} B_n(\eta)\int_0^{\tau} s^{\frac{1}{2}n}\,\mathrm{i}^n\mathrm{erfc}\left(\frac{\eta}{2s^{\frac{1}{2}}}\right)ds$$

$$= 2\pi^{\frac{1}{2}}\eta^{-1}A_1(\eta)\,\mathrm{erfc}\,\lambda - \pi^{\frac{1}{2}}\eta A_2(\eta)(\mathrm{erfc}\,\lambda - \pi^{-\frac{1}{2}}\lambda^{-1}e^{-\lambda^2}) +$$

$$+\sum_{n=0}^{\infty}(-)^n 2^{-n}\eta^{n+2}B_n(\eta)\frac{J_n}{(n+2)!}, \tag{4.13}$$

where $\lambda = \eta/2\tau^{\frac{1}{2}}$ and

$$J_n = (-)^n \tfrac{1}{2}(n+2)!\int_{\lambda}^{\infty} u^{-n-3}\,\mathrm{i}^n\mathrm{erfc}\,u\,du.$$

When $n \geqslant 1$ it is found by repeated integration by parts that

$$J_n = J_0 + \tfrac{1}{2}\sum_{r=1}^{n}(-)^r(r+1)!\,\lambda^{-r-2}\,\mathrm{i}^r\mathrm{erfc}\,\lambda. \tag{4.14}$$

Having determined $J_1, J_2, \ldots, J_5$ from (4.14) and having evaluated J_0 we

find that

$$\left.\begin{aligned}
J_0 &= (1+\tfrac{1}{2}\lambda^{-2})\operatorname{erfc}\lambda-\pi^{-\frac{1}{2}}e^{-\lambda^2}\lambda^{-1}\\
J_1 &= (1+\tfrac{3}{2}\lambda^{-2})\operatorname{erfc}\lambda-\pi^{-\frac{1}{2}}e^{-\lambda^2}(\lambda^{-1}+\lambda^{-3})\\
J_2 &= (1+3\lambda^{-2}+\tfrac{3}{4}\lambda^{-4})\operatorname{erfc}\lambda-\pi^{-\frac{1}{2}}e^{-\lambda^2}(\lambda^{-1}+\tfrac{5}{2}\lambda^{-3})\\
J_3 &= (1+5\lambda^{-2}+\tfrac{15}{4}\lambda^{-4})\operatorname{erfc}\lambda-\pi^{-\frac{1}{2}}e^{-\lambda^2}(\lambda^{-1}+\tfrac{9}{2}\lambda^{-3}+2\lambda^{-5})\\
J_4 &= (1+\tfrac{15}{2}\lambda^{-2}+\tfrac{45}{4}\lambda^{-4}+\tfrac{15}{8}\lambda^{-6})\operatorname{erfc}\lambda-\pi^{-\frac{1}{2}}e^{-\lambda^2}(\lambda^{-1}+7\lambda^{-3}+\tfrac{33}{4}\lambda^{-5})\\
J_5 &= (1+\tfrac{21}{2}\lambda^{-2}+\tfrac{105}{4}\lambda^{-4}+\tfrac{105}{8}\lambda^{-6})\operatorname{erfc}\lambda-\\
&\qquad -\pi^{-\frac{1}{2}}e^{-\lambda^2}(\lambda^{-1}+10\lambda^{-3}+\tfrac{87}{4}\lambda^{-5}+6\lambda^{-7})
\end{aligned}\right\}. \tag{4.15}$$

Hence the first eight terms in the expansion (4.13) have been determined, A_1, A_2, and B_n being defined by (4.9), the $a_{m,n}$ being given in Table 1 and the J_n being given by (4.15), λ denoting $\eta/2\tau^{\frac{1}{2}}$.

The corresponding skin friction is given by (4.11) and (4.12) and the integral in (4.12) vanishes since $F_w(\tau)$ and $F_w(\tau-s)$ are both unity throughout the range of integration. Hence

$$\left(\frac{\partial F}{\partial\eta}\right)_{\eta=0} = -\frac{1}{(\pi\tau)^{\frac{1}{2}}}+2a_{1,2}\left(\frac{\tau}{\pi}\right)^{\frac{1}{2}}+\sum_{n=0}^{\infty}\frac{a_{1,n+3}\,\tau^{1+\frac{1}{2}n}}{\Gamma(2+\frac{1}{2}n)}. \tag{4.16}$$

4.2. *Solution for large times*

We now expand $\bar{f}(\eta,p)$ in a series for small p and invert term by term to obtain a formal solution for large τ (Carslaw and Jaeger (7)). Glauert (3) has obtained a solution for small values of p in the form

$$\bar{f}(\eta,p) = \sum_{n=0}^{\infty}p^n f_n(\eta), \tag{4.17}$$

where

$$\left.\begin{aligned}
f_0(\eta) &= \phi''(\eta)/\phi''(0)\\
f_n(\eta) &= -\phi''\left[\int_0^{\eta}\phi''e^{\phi^0}If_{n-1}\,d\eta+I\int^{\infty}\phi''e^{\phi^0}f_{n-1}\,d\eta\right]\\
\phi^0 &= \int_0^{\eta}\phi\,d\eta\\
I &= \int_0^{\eta}e^{-\phi^0}/\phi''^2\,d\eta
\end{aligned}\right\}. \tag{4.18}$$

On inverting his solution (4.17) term by term, we obtain the result

$$f(\eta,\tau) = \sum_{n=0}^{\infty}f_n(\eta)\,\frac{d^n\delta(\tau)}{d\tau^n}, \tag{4.19}$$

where $\delta(\tau)$ is the delta function. Then $F(\eta,\tau)$ is given by (4.10) so that

$$F(\eta,\tau) = \int_0^\tau F_w(\tau-s) \sum_{n=0}^{\infty} f_n(\eta) \frac{d^n\delta(s)}{ds^n}\, ds$$

$$= \sum_{n=0}^{\infty} f_n(\eta) \int_0^\tau F_w(\tau-s) \frac{d^n\delta(s)}{ds^n}\, ds = \sum_{n=0}^{\infty} f_n(\eta) \frac{d^n F_w(\tau)}{d\tau^n}, \quad (4.20)$$

by successive integration by parts, and consequently the skin friction is given by equation (4.11) with

$$\left(\frac{\partial F}{\partial \eta}\right)_{\eta=0} = \sum_{n=0}^{\infty} f_n'(0) \frac{d^n F_w(\tau)}{d\tau^n}. \quad (4.21)$$

In the particular case of a wall set impulsively in uniform motion from rest, for which $F_w(\tau) = H(\tau)$, the unit function, equations (4.20), (4.21) reduce to

$$\left.\begin{aligned} F(\eta,\tau) &= f_0(\eta) \\ \left(\frac{\partial F}{\partial \eta}\right)_{\eta=0} &= f_0'(0) \end{aligned}\right\} \quad (4.22)$$

for large τ, which agrees with Glauert's result for the wall in steady motion as we would expect. The value of $f_0'(0)$ in (4.22) was given by Glauert, and is

$$f_0'(0) = -0{\cdot}8113. \quad (4.23)$$

4.3. *Summary of Laplace-transform solutions*

In sections 4.1 and 4.2, we have obtained a formal solution for the unsteady velocity distribution (3.4) which is given by (4.10), (4.9) for small times and by (4.18) and (4.20) for large times, $\epsilon F_w(\tau)$ being the velocity of the wall. The skin friction is given by (4.13) and (4.15) for small times and by (4.11) and (4.21) for large times.

In the particular case of a wall at rest, impulsively set in uniform motion at $\tau = 0$ with velocity ϵ, the velocity for small times is determined from (4.13) to (4.15) while for large times it is determined from (4.22). The unsteady skin friction is proportional to $-(\partial F/\partial\eta)_{\eta=0}$, which is plotted in Fig. 1. It is given by (4.16) for $\tau \leqslant 1$ and the limiting value as $\tau \to \infty$ is given by (4.22) and (4.23). It is readily seen that for small times the curve very rapidly approaches the limiting value; so this curve may be extrapolated to its limiting value to a high degree of accuracy. Hence, for this particular flow, the skin friction has been obtained with sufficient accuracy by use of the Laplace-transform solutions only. In general, however, the curve for small times may not link up with the limiting value for large times. For this reason an alternative approximate method is

given in section 5, which may be used to achieve an optimum join between the solutions for small and large times.

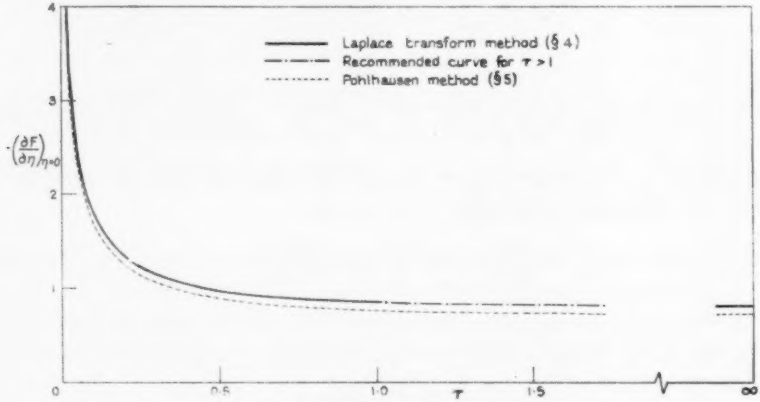

FIG. 1. Variation of unsteady part of skin friction with time.

It is readily shown that (4.10) formally satisfies (3.5) provided that certain differential equations are satisfied by the functions $A_1(\eta)$, $A_2(\eta)$, $B_n(\eta)$ of (4.9). It follows from these differential equations that, in place of the power series definitions, these functions may be written as functions of ϕ which are more amenable to calculation for values of η which are not small. We find

$$\left.\begin{aligned} A_1 &= \frac{\eta}{2\pi^{\frac{1}{2}}}\exp\Bigl\{-\tfrac{1}{2}\int_0^\eta \phi\,d\eta\Bigr\} \\ A_2 &= -\frac{1}{8\pi^{\frac{1}{2}}}\Bigl(6\phi+\int_0^\eta \phi^2\,d\eta\Bigr)\exp\Bigl\{-\tfrac{1}{2}\int_0^\eta \phi\,d\eta\Bigr\} \\ B_0 &= \tfrac{1}{32}\Bigl[-12\phi'+7\phi^2+3\phi\int_0^\eta \phi^2\,d\eta+\tfrac{1}{4}\Bigl(\int_0^\eta \phi^2\,d\eta\Bigr)^2\Bigr]\exp\Bigl\{-\tfrac{1}{2}\int_0^\eta \phi\,d\eta\Bigr\} \end{aligned}\right\}, \tag{4.24}$$

and the rest are determined successively by the recurrence relation

$$B_{n+1}\exp\Bigl\{\tfrac{1}{2}\int_0^\eta \phi\,d\eta\Bigr\} = -B_n'(0)+(B_n'+\tfrac{1}{2}\phi B_n)\exp\Bigl\{\tfrac{1}{2}\int_0^\eta \phi\,d\eta\Bigr\}- \\ -\int_0^\eta (\tfrac{3}{2}\phi'+\tfrac{1}{4}\phi^2)B_n\exp\Bigl\{\tfrac{1}{2}\int_0^\eta \phi\,d\eta\Bigr\}\,d\eta. \tag{4.25}$$

In fact they are all of the form of the product of $\exp\Bigl\{-\tfrac{1}{2}\int_0^\eta \phi\,d\eta\Bigr\}$ with

a function of ϕ which tends to infinity as $\eta \to \infty$ like a power of η. A rigorous justification that (4.10) satisfies (3.5) presents difficulties arising from the lack of knowledge of the properties of $B_n(\eta)$ for large n. However, it appears from the case of an impulse, that the Laplace-transform method for small times yields the exact solution. Similar comments can be made for the large times result, (4.20).

For the impulse case we define a time, τ_c, which is characteristic of the time taken for the flow to settle down into its new steady state and may be defined as follows. Let τ_s be the steady limiting value of the unsteady skin friction. Then the characteristic time, τ_c, is defined by

$$\tau_c = \int_0^\infty \left(\frac{\tau_w - \tau_{w_0}}{\tau_s} - 1\right) d\tau, \tag{4.26}$$

and this integral may be evaluated numerically to yield

$$\tau_c = 0{\cdot}60(8). \tag{4.27}$$

The time taken for the flow to settle down to its ultimate state will be about three times this. Since $\tau = at$ and the undisturbed displacement thickness, δ_{10}, is given by

$$(a/\nu)^{\frac{1}{2}}\delta_{10} = \int_0^\infty (1-\phi')\, d\eta = 0{\cdot}6479,$$

it follows that $$\nu t_c/\delta_{10}^2 = \nu\tau_c/a\delta_{10}^2 = 1{\cdot}4(5). \tag{4.28}$$

It is noteworthy that this dimensionless characteristic time is of the same order of magnitude as the characteristic time for the asymptotic suction profile (Watson (**5**)). In that case $\nu t_c/\delta_{10}^2$ was unity. For a boundary layer of displacement thickness 0·1 cm, in air, t_c is here about 0·1 sec.

5. Alternative approximate method of solution

We now obtain an approximate solution of (3.5), (3.6) by a Pohlhausen type of method. In this we satisfy the differential equation (3.5) only at the wall, at infinity, and on an average by integrating it across the boundary layer. When integrated with respect to η from 0 to ∞, equation (3.5) becomes

$$\int_0^\infty \frac{\partial F}{\partial \tau}\, d\eta = -(F')_{\eta=0} - 2\int_0^\infty \phi' F\, d\eta \tag{5.1}$$

where use has been made of the boundary conditions (3.2) and (3.6). We now approximate to F by means of a cubic in η/Δ,

and by
$$\left.\begin{aligned} F &= A + B(\eta/\Delta) + C(\eta/\Delta)^2 + D(\eta/\Delta)^3 \quad &\text{for } 0 \leqslant \eta < \Delta \\ F &= 0 &\text{for } \eta \geqslant \Delta \end{aligned}\right\}, \tag{5.2}$$

where A, B, C, D are chosen to satisfy the four boundary conditions

$$\left.\begin{aligned} F(0,\tau) &= F_w(\tau) & F(\Delta,\tau) &= 0 \\ F''(0,\tau) &= \frac{dF_w(\tau)}{d\tau} & F'(\Delta,\tau) &= 0 \end{aligned}\right\}. \tag{5.3}$$

These conditions lead to

$$F(\eta,\tau) = \tfrac{1}{2}F_w\{2-3(\eta/\Delta)+(\eta/\Delta)^3\}-\tfrac{1}{4}\Delta^2\frac{dF_w}{d\tau}\{(\eta/\Delta)-2(\eta/\Delta)^2+(\eta/\Delta)^3\} \tag{5.4}$$

for $0 \leqslant \eta < \Delta$. We note that this is the typical single-parameter velocity distribution which arises in Pohlhausen treatments, the parameter, $\left(\frac{\Delta^2}{F_w}\right)\frac{dF_w}{d\tau}$, and the Pohlhausen parameter, $\Lambda = \left(\frac{\delta^2}{\nu}\right)\frac{du_1}{dx}$, being similar in form. We now substitute (5.2) and (5.4) into (5.1) and obtain the ordinary differential equation for Δ,

$$3\left(6F_w-\Delta^2\frac{dF_w}{d\tau}\right)\frac{d\Delta}{d\tau} = \left(\frac{72}{\Delta}F_w-6\Delta\frac{dF_w}{d\tau}+\Delta^3\frac{d^2F_w}{d\tau^2}\right)-$$
$$-96\int\limits_0^\Delta\left[\tfrac{1}{2}F_w\{2-3(\eta/\Delta)+(\eta/\Delta)^3\}-\tfrac{1}{4}\Delta^2\frac{dF_w}{d\tau}\{(n/\Delta)-2(\eta/\Delta)^2+(\eta/\Delta)^3\}\right]\phi'\,d\eta, \tag{5.5}$$

which can be solved by standard methods. Having determined Δ as a function of τ, we find from (5.4) that, for $\tau > 0$,

$$-\left(\frac{\partial F}{\partial\eta}\right)_{\eta=0} = \frac{3F_w}{2\Delta}+\left(\frac{\Delta}{4}\right)\frac{dF_w}{d\tau}, \tag{5.6}$$

a known function of τ. The skin friction follows from (4.13).

We now give details for the particular case considered earlier, namely when the wall at rest is impulsively set in uniform motion at $\tau = 0$, so that $F = 0$ for $\tau < 0$ and $F_w(\tau)$ is constant for $\tau > 0$. Then, for $\tau > 0$, (5.5) reduces to

$$\frac{1}{4}\frac{d\Delta}{d\tau} = \frac{1}{\Delta}-\frac{4}{3}\int\limits_0^\Delta\{1-\tfrac{3}{2}(\eta/\Delta)+\tfrac{1}{2}(\eta/\Delta)^3\}\,\phi'\,d\eta. \tag{5.7}$$

We may further approximate to the steady velocity distribution,

$$u_0/U_0 = \phi'(\eta),$$

by its Pohlhausen value. Hence, following Goldstein ((**8**), p. 156),

$$\left.\begin{aligned} \phi' &= 1-(1-\eta^*)^3\{1+(1-\tfrac{1}{6}\Lambda)\eta^*\} && \text{for } \eta^* < 1 \\ \phi' &= 1 && \text{for } \eta^* \geqslant 1 \end{aligned}\right\}, \tag{5.8}$$

where $\quad \eta^* = z/\delta = \eta/\Lambda^{\frac{1}{2}}, \qquad \Lambda = (\delta^2/\nu)\,dU_0/dx = \delta^2 a/\nu \qquad$ (5.9)

and for stagnation point flow, $\Lambda = 7{\cdot}05$. We assume, as is reasonable physically, that the unsteady boundary-layer thickness, $(\nu/a)^{\frac{1}{2}}\Delta$, will not exceed the steady boundary-layer thickness, δ, that is, $(\nu/a)^{\frac{1}{2}}\Delta \leqslant \delta$ or $\Delta \leqslant \Lambda^{\frac{1}{2}} = (7{\cdot}05)^{\frac{1}{2}}$ for all values of τ. This has been verified *a posteriori* because Δ is found to vary monotonically between 0 and $2{\cdot}079$. It follows that we may use the quartic expression (5.8) for ϕ' throughout the unsteady boundary layer. Then on substituting into (5.7) and integrating we obtain an ordinary differential equation which integrates immediately to give

$$4\tau = \int_0^\Delta \frac{\Delta\, d\Delta}{\left[1 - \frac{4}{3}\Delta\left\{\frac{1}{10}\left(2+\frac{\Lambda}{6}\right)\frac{\Delta^2}{\Lambda^{\frac{1}{2}}} - \frac{\Delta^3}{48} - \frac{3}{140}\left(2-\frac{\Lambda}{2}\right)\frac{\Delta^4}{\Lambda^{\frac{1}{2}}} + \frac{1}{80}\left(1-\frac{\Lambda}{6}\right)\frac{\Delta^5}{\Lambda^2}\right\}\right]}, \tag{5.10}$$

the limits of integration being chosen so that $\Delta = 0$ when $\tau = 0$. The integrand of (5.10) may be split into partial fractions and integrated exactly, yielding an analytic function τ of Δ, or Δ may be found as a function of τ from (5.10) by numerical integration. The quantity $\left(\frac{\partial F}{\partial \eta}\right)_{\eta=0}$ then follows from (5.6) with $F_w = 1$, namely

$$-\left(\frac{\partial F}{\partial \eta}\right)_{\eta=0} = \frac{3}{2\Delta}. \tag{5.11}$$

This is plotted in Fig. 1.

We note that the curve for the unsteady skin friction as determined by the Pohlhausen method shows good agreement with those obtained by the Laplace-transform method for small and large times. In general the former will enable an interpolation to be made in any range of values of τ for which the latter curves are not valid, though clearly in the present case satisfactory interpolation is possible without the use of the Pohlhausen method.

It can readily be shown that, for small times, the unsteady skin friction as given by the Pohlhausen treatment has the correct form (in the sense that it varies like $\tau^{-\frac{1}{2}}$) but is 6 per cent too low compared with the leading term of the exact expansion (4.16), whereas at large times the limiting value is 11 per cent too low.

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THE ELLIPTIC CYLINDER IN A SHEAR FLOW WITH HYPERBOLIC VELOCITY PROFILE

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SUMMARY

The stream function for the shear flow with hyperbolic velocity profile past an elliptic cylinder has been determined as an infinite series of Mathieu functions. It is found that the stagnation streamline of the flow is displaced towards a region of higher velocity, this displacement increasing (i) with increase of angle of incidence of the cylinder to the main stream, (ii) as the stream becomes progressively non-uniform, (iii) with increase of minor axis length when the major axis length remains invariant. In each case the displacement reaches a limiting value as the cylinder moves away from the axis of symmetry of the stream. These limiting values are reached at critical distances from the axis of symmetry, which decrease as the stream becomes progressively non-uniform, but these distances are approximately independent of incidence.

The pressure coefficients and the resultant force and moment coefficients associated with the cylinder have also been obtained, and investigated numerically for the flat plate type of cylinder.

1. Introduction

WHEN the vorticity is constant throughout the fluid, the characteristics of the flow in the presence of cylindrical bodies of simple cross-sections have been investigated by Tsien (**1**), James (**2**), and Mitchell and Murray (**3**). A close investigation of the stagnation points in the field of flow was carried out in (**1**) and (**3**), and in particular in (**3**) the displacement of the stagnation streamline at large distances from the cylinder was determined. Considerable interest has for some time been concentrated on the pitot-tube effect concerning the displacement of the stagnation streamline by a pitot-tube set in a shear flow. Although this is strictly a three-dimensional effect as shown by Hall (**4**) and Lighthill (**5**), the two-dimensional approach of Murray and Mitchell (**6**) as applied to a circular cylinder representation of the pitot-tube confirms the main experimental result due to Young and Maas (**7**), viz. that the displacement of the streamline is towards a region of higher velocity. However, it is found that the two-dimensional theory underestimates the magnitude of the displacement as compared with experimental results.

The flow of fluid past a thin aerofoil in a shear flow with parabolic velocity profile was previously investigated by the author (**8**) with the main object of determining the hydrodynamical forces acting on the

aerofoil. The solution was an approximate one, the results being applicable only when the aerofoil was in the vicinity of the axis of symmetry of the field of flow. The present study is an extension which gives an exact treatment for a more general type of shear flow past an elliptic cylinder and a flat plate, and thus enlarges on that covered by (**6**). Since the subject-matter of this paper was completed a further paper by Murray (**9**) has appeared which investigates the non-uniform shear flow of a fluid past cylinders with sections of a general shape, the elliptic cylinder being a particular example. However, the analysis is complicated and the determination of certain coefficients tedious to carry out, even in the case considered when the cylinder has its axes parallel to or perpendicular to the direction of flow. It was noted, however, by Murray that an analysis involving Mathieu functions was possible, although this was not developed.

In view of the fact that the analysis of this paper involves Mathieu functions many references are made to the treatise of McLachlan dealing with these functions, (**10**). The numerical analysis has been based on the N.B.S. tables (**11**), with reference also to Watson's treatise on Bessel functions (**12**).

2. The stream function

The fluid is assumed to be incompressible and inviscid, and the flow is two-dimensional, steady and rotational, i.e. it is associated with a vorticity which in general varies with position in the fluid. Attention is confined to any one plane of flow and a set of orthogonal Cartesian axes $O'X$, $O'Y$ locate position in this plane. The fluid velocity has components (v_X, v_Y) relative to these stream axes, and the continuity equation can be satisfied by introducing a stream function $\psi(X, Y)$, where

$$v_X = \partial\psi/\partial Y, \qquad v_Y = -\partial\psi/\partial X.$$

The vorticity vector has its non-zero component directed normal to the plane and is of magnitude

$$\omega = \frac{\partial v_Y}{\partial X} - \frac{\partial v_X}{\partial Y} = -\nabla^2\psi. \tag{1}$$

In steady motion the vorticity is constant along a streamline, implying $\nabla^2\psi = f(\psi)$, where $f(\psi)$ is an arbitrary function of ψ. Following the method of (**6**) and (**9**) a possible rotational flow is defined by choosing $f(\psi) = \psi/c^2$, where c is an arbitrary standard of length, thus leading to the flow equation

$$\nabla^2\psi = \psi/c^2. \tag{2}$$

In the absence of the cylinder the undisturbed stream is assumed to be defined by a stream function $\psi_0(X, Y)$, the vorticity in the stream being defined by (1) and the flow equation given by (2), with ψ replaced by ψ_0.

It is assumed that the streamlines of the undisturbed flow are all parallel to the axis $O'X$, hence ψ_0 is a function of Y only, and the flow equation can be written as $d^2\psi_0/dY^2 = \psi_0/c^2$, with a solution

$$\psi_0 = -cV\left(\sinh\frac{Y}{c}+N\cosh\frac{Y}{c}\right). \tag{3}$$

Here V is the stream velocity along the negative X-axis, and $N = c\omega_0/V$ is a dimensionless constant, where ω_0 is the vorticity component of the flow on the X-axis. If $N = 0$, then the flow is symmetrical about the X-axis, the velocity profile being of the hyperbolic cosine form, with velocity V along the negative X-axis and zero vorticity on this axis. The length c is a parameter indicating the deviation of the stream from the uniform. The stream is similar to that which occurs behind a symmetrical obstacle set in a uniform flow, and will be the main subject of the present investigation.

Ultimately a cylinder is to be introduced into the field of flow with generators perpendicular to the flow plane. The right section of the cylinder has an elliptic profile, and the cylinder axes Ox, Oy are taken respectively along the major and minor axes of the ellipse with O at its centre. The stream function for the disturbed flow past the cylinder is $\psi(x,y)$, where $v_x = \partial\psi/\partial y$, $v_y = -\partial\psi/\partial x$ are the velocity components in the direction of the cylinder axes. As for the undisturbed stream it follows that (2) is satisfied by the stream function ψ in terms of x and y. If O has coordinates (X_0, Y_0) relative to the stream axes, and the positive x-axis is inclined at an angle θ to the positive X-axis, then

$$Y = Y_0+x\sin\theta+y\cos\theta. \tag{4}$$

When the cylinder is introduced in the stream the stream function ψ_0 is modified by an amount Ψ, becoming $\psi = \psi_0+\Psi$. Since both ψ_0 and ψ satisfy a flow equation similar to (2), then

$$\nabla^2\Psi = \Psi/c^2. \tag{5}$$

It is noticed that when $\Psi \to 0$, then $\nabla^2\psi \to \nabla^2\psi_0$. This verifies that at large distances from the cylinder where the latter has little effect the disturbed stream has the same hydrodynamical characteristics as the undisturbed stream.

The elliptic profile of the cylinder section is assumed to have a semi-major axis of length $u = h\cosh\xi_0$ and a semi-minor axis of length $h\sinh\xi_0$, where $2h$ is the focal distance of the ellipse, and $\xi_0 \geqslant 0$. The natural coordinates for flow past the ellipse are the elliptic coordinates (ξ, η), where

$$x = h\cosh\xi\cos\eta, \qquad y = h\sinh\xi\sin\eta, \tag{6}$$

with $\xi = \xi_0$ on the ellipse, and $0 \leqslant \eta \leqslant 2\pi$.

With this change of coordinates, (5) transforms to

$$\frac{\partial^2\Psi}{\partial\xi^2}+\frac{\partial^2\Psi}{\partial\eta^2} = 2q(\cosh 2\xi-\cos 2\eta)\Psi,$$

with $q = k^2 = h^2/4c^2$, and if the function Ψ is expressed in the form $\Psi = f_1(\eta)f_2(\xi)$, then on separating the variables there results

$$\frac{d^2f_1}{d\eta^2}+(a+2q\cos 2\eta)f_1 = 0, \tag{7}$$

$$\frac{d^2f_2}{d\xi^2}-(a+2q\cosh 2\xi)f_2 = 0, \tag{8}$$

where a is the separation constant. These equations are respectively the ordinary and modified Mathieu differential equations (**10**). It is necessary that $\Psi(\xi,\eta)$ be periodic in η with period 2π, and that it must tend to zero as $\xi \to +\infty$, i.e. at large distances from the cylinder, where $\psi \to \psi_0$. These conditions are satisfied by a suitable choice of $f_1(\eta)$ and $f_2(\xi)$. The differential equations (7) and (8) have solutions which satisfy the required conditions for a discrete set of eigenvalues of the constant a.

The functions $f_1(\eta)$ are then the ordinary Mathieu functions of integral order denoted by $\mathrm{ce}_m(\eta,-q)$ and $\mathrm{se}_m(\eta,-q)$ with $m \geqslant 0$ (**10**, p. 21). These are continuous functions for all real η, and their series representations are absolutely and uniformly convergent.

Solutions for $f_2(\xi)$ take the form $\mathrm{Fek}_m(\xi,-q)$ and $\mathrm{Gek}_m(\xi,-q)$, which are the modified Mathieu functions of integral order (**10**, p. 248). These functions converge rapidly and monotonically to zero as $\xi \to +\infty$, the rate of convergence increasing with increase of ξ. The series representations converge uniformly in any finite part of the ξ-plane, including $\xi = 0$.

The two sets of functions form solution pairs (**10**, p. 174), the complete solution of (5) in terms of elliptic coordinates being

$$\Psi/cV = \sum_{m=0}^{\infty}\{D_m\,\mathrm{Fek}_m(\xi,-q)\mathrm{ce}_m(\eta,-q)+E_m\,\mathrm{Gek}_m(\xi,-q)\mathrm{se}_m(\eta,-q)\}, \tag{9}$$

where D_m and E_m are dimensionless arbitrary constants, with $E_0 = 0$.

3. Determination of the coefficients

The coefficients D_m and E_m for $m = 0, 1,\ldots$, can be determined by imposing the boundary conditions at the surface of the cylinder, viz. that the profile of the cylinder section is a streamline of flow. Mathematically it is necessary that $\psi/cV = p$, a constant, when $\xi = \xi_0$ for all η in the range $0 \leqslant \eta \leqslant 2\pi$. Now $k = h/2c$, and if we write

$$w = \cosh\xi\cos\eta\sin\theta+\sinh\xi\sin\eta\cos\theta,$$

then in terms of elliptic coordinates, with Y defined by (4), the undisturbed stream function ψ_0 takes the form

$$\psi_0 = -cV\{\sinh(b+2kw)+N\cosh(b+2kw)\}, \tag{10}$$

where $b = Y_0/c$. Ultimately it will be necessary to express quantities in terms of the ratio $L = Y_0/u$, i.e. $L = b\beta$, where $\beta = c/u$. All physical quantities for the flow past the cylinder can thus be expressed in terms of the non-dimensional quantities N, q $(= k^2)$, β, and L (or b). It is noted that $\xi_0 = \operatorname{sech}^{-1}(2k\beta)$. The boundary condition on the cylinder requires that $\psi_0+\Psi = cVp$, when $\xi = \xi_0$, or from (9) and (10),

$$p+\sinh(b+2kw_0)+N\cosh(b+2kw_0)$$
$$= \sum_{m=0}^{\infty}\{D_m \operatorname{Fek}_m(\xi_0,-q)\operatorname{ce}_m(\eta,-q)+E_m \operatorname{Gek}_m(\xi_0,-q)\operatorname{se}_m(\eta,-q)\}, \tag{11}$$

where $w_0 = \{w\}_{\xi=\xi_0}$, and is to be satisfied for all η in the range $0 \leqslant \eta \leqslant 2\pi$.

It is possible to determine D_m and E_m as functions of p by using the normalized orthogonality relations for the ordinary Mathieu functions (**10**, p. 24), which are

$$\int_0^{2\pi} \operatorname{ce}_m(\eta,-q)\operatorname{ce}_n(\eta,-q)\,d\eta = \begin{cases}\pi & (m=n),\\ 0 & (m\neq n);\end{cases}$$

$$\int_0^{2\pi} \operatorname{se}_m(\eta,-q)\operatorname{se}_n(\eta,-q)\,d\eta = \begin{cases}\pi & (m=n),\\ 0 & (m\neq n);\end{cases}$$

$$\int_0^{2\pi} \operatorname{ce}_m(\eta,-q)\operatorname{se}_n(\eta,-q)\,d\eta = 0 \quad (\text{all } m,\, n).$$

On applying these relations to (11), it can be shown that

$$D_{2n}\operatorname{Fek}_{2n}(\xi_0,-q) = 2(-1)^n pA_0^{2n}+2(\sinh b+N\cosh b)C_{2n}(k), \tag{12}$$

$$D_{2n+1}\operatorname{Fek}_{2n+1}(\xi_0,-q) = 2(\cosh b+N\sinh b)C_{2n+1}(k), \tag{13}$$

$$E_{2n+2}\operatorname{Gek}_{2n+2}(\xi_0,-q) = 2(\sinh b+N\cosh b)S_{2n+2}(k), \tag{14}$$

$$E_{2n+1}\operatorname{Gek}_{2n+1}(\xi_0,-q) = 2(\cosh b+N\sinh b)S_{2n+1}(k), \tag{15}$$

where

$$C_m(k) = (-1)^m C_m(-k) = \frac{1}{2\pi}\int_0^{2\pi} e^{2kw_0}\operatorname{ce}_m(\eta,-q)\,d\eta,$$

$$S_m(k) = (-1)^m S_m(-k) = \frac{1}{2\pi}\int_0^{2\pi} e^{2kw_0}\operatorname{se}_m(\eta,-q)\,d\eta.$$

Expressions for these integrals are determined in the appendix.

Equations (12)–(15) determine D_{2n} in terms of p, and D_{2n+1}, E_{2n+2},

E_{2n+1}, in terms of known functions. If p is known the function Ψ, given by (9), and thus $\psi = \psi_0 + \Psi$, are known explicitly. The flow pattern is thus uniquely determined.

The boundary condition already imposed on the stream function is not sufficient to determine the constant p. The ideal state of affairs would be to determine the lift force on the cylinder theoretically in terms of p and compare this with experimental values for the lift, thus determining a value or range of values for p. In order to obtain a theoretical result to the problem recourse can be made in the case of the elliptic cylinder to the assumption that the circulation round the curve coinciding with the section profile of the cylinder in the disturbed stream has the same value as that round the same curve in the undisturbed stream, cf. Tsien (1). It must be pointed out, however, that this extra condition, although a plausible assumption, has been introduced in order to provide a solution to the problem, and in fact requires experimental justification.

If s is the curved profile of the cylinder section then the Tsien condition becomes

$$\int_s (\partial\Psi/\partial\xi)_{\xi=\xi_0}\, d\eta = 0. \tag{16}$$

From the definitions of the ordinary Mathieu functions,

$$\int_0^{2\pi} \mathrm{ce}_m(\eta, -q)\, d\eta = \begin{cases} 2\pi(-1)^n A_0^{2n} & (m = 2n), \\ 0 & (m = 2n+1) \end{cases}$$

and

$$\int_0^{2\pi} \mathrm{se}_m(\eta, -q)\, d\eta = 0 \quad (\text{all } m).$$

Hence substituting for Ψ from (9) into (16), it follows that

$$\sum_{m=0}^{\infty} (-1)^n D_{2n} A_0^{2n}\, \mathrm{Fek}'_{2n}(\xi_0, -q) = 0, \tag{17}$$

where ($'$) signifies differentiation with respect to ξ. The constant p is now determined by substituting for D_{2n} from (12) into (17), leading to the result

$$p = -K(\sinh b + N\cosh b),$$

where

$$K(\xi_0, \theta, q) = \frac{\sum_{n=0}^{\infty} (-1)^n A_0^{2n} F_{2n}(\xi_0, -q) C_{2n}(k)}{\sum_{n=0}^{\infty} (A_0^{2n})^2 F_{2n}(\xi_0, -q)} \tag{18}$$

and by definition

$$F_{2n}(\xi_0, -q) = \mathrm{Fek}'_{2n}(\xi_0, -q)/\mathrm{Fek}_{2n}(\xi_0, -q).$$

Substitution for p into (12) gives the final form of D_{2n}. If this expression for D_{2n} and the coefficients in (13), (14), and (15) are substituted into (9)

this yields the final expression for Ψ, and with ψ_0 given by (3) the stream function ψ of the flow past the elliptic cylinder.

It is possible to approximate to the parameter K in (18), and thus to the stream function ψ, for streams with small velocity gradients. In this case the stream is nearly uniform, and then $\beta = c/u$ is large, or q is small.

The number q is limited such that $1 > |q \log q| > q$. Then it is known that $A_{2r}^{2n} \simeq O(q^{|n-r|})$ (**10**, p. 46; Errata p. xii), and from (**10**, p. 382) it can be shown that $F_{2n}(\xi_0, -q)/F_0(\xi_0, -q)$ is $O(\log q)$. Again from (**10**, p. 46) it is seen that $A_0^2 = \frac{1}{4}q + O(q^3)$, $A_2^0 = -\frac{1}{2}qA_0^0 + O(q^3)$, and using the series forms for the modified Mathieu and Bessel functions it can be proved that

$$\frac{F_2(\xi_0, -q)}{F_0(\xi_0, -q)} = -\frac{2(\frac{1}{2}\log q + \gamma - \log 2 + \xi_0)}{1 + \frac{1}{2}q(\log q)\sinh 2\xi_0} + O(q \log q),$$

where γ is the Euler constant.

On using the result $I_{2r}(z) \simeq O(q^r)$, it can readily be shown that (18) reduces to

$$\begin{aligned} K = {} & 1 + \tfrac{1}{2}q(\cosh 2\xi_0 - \cos 2\theta) + \frac{q(S-1)}{4S}(\operatorname{cosech} 2\xi_0 - \coth 2\xi_0 \cos 2\theta) + \\ & + \tfrac{1}{16}q^2\{(\cosh 2\xi_0 - \cos 2\theta)^2 + 2(1 - 2\mu + 2\xi_0)(1 - \cosh 2\xi_0 \cos 2\theta)\} + \\ & + O(q^3 \log q), \end{aligned} \tag{19}$$

where $\mu = \log 2 - \gamma = 0{\cdot}1159\ldots$, and $S = 1 + \frac{1}{2}q(\log q)\sinh 2\xi_0$.

These results simplify in two limiting cases:

(i) If $h \to 0$ and $\xi_0 \to \infty$ simultaneously such that $he^{\xi_0} \to 2R$, then the ellipse degenerates into the circle of radius R (**10**, pp. 367–9). It can readily be shown from (18) that in general $K \to I_0(R/c)$, a modified function of the first kind, and that ψ reduces to that deduced in (**6**). In particular for the near-uniform stream, $K = 1 + R^2/4c^2 + O\{(R/c)^4\}$.

(ii) If $\xi_0 \to 0$ then the cylinder takes the form of a flat plate, and from (A 1) of the appendix, $z = 2k \sin\theta$ and $\delta = \frac{1}{2}\pi$. It follows from (A 5) and (A 6) that $S_m(k) = 0$ for $m \geqslant 1$, and from (14) and (15) it is seen that $E_m = 0$ for $m \geqslant 1$ in (9). For the flat plate of length $2u$, with $\xi_0 = 0$, (19) becomes

$$K = 1 + q\sin^2\theta + \tfrac{1}{4}q^2(\log q)\sin^2\theta + \tfrac{1}{4}q^2(1 - 2\mu + \sin^2\theta)\sin^2\theta + O(q^3 \log q).$$

In this case of the flat plate the stream velocities at both leading and trailing edges become infinite in magnitude. This means that in the vicinity of these edges the stream function as deduced using the Tsien condition will not define a real fluid flow.

4. Displacement of the stagnation streamline

The displacement of the stagnation streamline from the cylinder for the general type of stream with hyperbolic velocity profile can be determined

by use of the stream function given by (9). It is then possible to determine the effect of cylinder shape, cylinder incidence, and form of the stream on the displacement of the stagnation streamline. This streamline curves off to infinity when it leaves the cylinder, its equation being $\psi(\xi,\eta) = cpV$. At large distances from the cylinder $\psi \to \psi_0$, hence if the stagnation streamline there has a displacement from the X-axis of magnitude d, its equation from (3) is given by

$$\sinh\frac{d}{c}+N\cosh\frac{d}{c} = K(\sinh b+N\cosh b). \tag{20}$$

On solving this equation we find

$$(N+1)\frac{d}{c} = \log[K(\sinh b+N\cosh b)+\{K^2(\sinh b+N\cosh b)^2+1-N^2\}^{\frac{1}{2}}]. \tag{21}$$

Since the numerical analysis which follows refers to cylinders of invariant major-axis length, and the stream form is allowed to vary, it is more convenient to express the displacement of the stagnation streamline as a multiple of the semi-major axis length of the elliptic section of the cylinder, viz. u. If the displacement of the streamline is measured from a line through O, the centre of the cylinder, parallel to the X-axis, i.e. the line $Y = Y_0$, then the parameter determining its magnitude is

$$D = (d-Y_0)/u, \quad \text{or} \quad D = \beta\left(\frac{d}{c}-b\right),$$

where $\beta = c/u$, and d/c is given by (21).

When the stream is symmetrical about the X-axis, with $N = 0$, then from (21),

$$D = \beta\sinh^{-1}\left(K\sinh\frac{L}{\beta}\right)-L, \tag{22}$$

where $L = Y_0/u = \beta b$. It is noticed from (22) that $D(L) = -D(-L)$, hence at mirror image points in the X-axis the displacements are equal in magnitude but oppositely directed. Also from (18) it can be shown that $K(\theta) = K(\pi-\theta)$, hence from (22) it is seen that $D(\theta) = D(\pi-\theta)$, and it is thus necessary to consider incidence only in the range $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. When the cylinder is at large distances from the X-axis, i.e. as $L \to \infty$, then the streamline has a limiting displacement given by $D_L = \beta\log K$. Also when the centre of the cylinder is on the X-axis, i.e. $L = 0$, then $T_0 = (dD/dL)_{L=0} = K-1$. Other results of interest are (i) for given β and L, $dD/dK \gtrless 0$ according as $L \gtrless 0$, i.e. at a given position of the cylinder away from $Y = 0$ in a given stream, D increases with K, (ii) if $dK/d\theta > 0$, $dD/d\theta \gtrless 0$ according as $L \gtrless 0$, indicating how the displacement varies with incidence at a given position in the stream, (iii) for

given β and K, $dD/dL \gtrless 0$ according as $K \gtrless 1$, also $dD/dL \to 0$ as $L \to \infty$, which indicates how the displacement varies with position of the cylinder in a given stream.

In order to indicate the order of magnitude of D, the particular example of an elliptic cylinder in a symmetrical stream is considered. If the minor axis of the ellipse is half the length of the major axis, then $\tanh \xi_0 = \frac{1}{2}$ and $h = \frac{1}{2}\sqrt{3}u$. The stream parameter $\beta = \frac{1}{4}\sqrt{3}$, hence $q = 1$, and the stream has large shear. Values of K are determined from (18) with $\theta = 0$, $\frac{1}{4}\pi$, $\frac{1}{2}\pi$. The functions $F_{2n}(\xi_0, -q)$ are derived from tables (**11**), and the $C_{2n}(k)$ as given in the appendix, the convergence being sufficiently rapid for computational purposes. It can be shown that

$$K = I_0(z)+0\cdot15754\, I_2(z)\cos 2\delta - 0\cdot00230\, I_4(z)\cos 4\delta - 0\cdot00006\, I_6(z)\cos 6\delta + \ldots$$

where $\tan \delta = 2\tan\theta$, $z^2 = \frac{4}{3}(5-3\cos 2\theta)$, and I_0, I_2,... are modified Bessel functions of the first kind. Table 1 gives values indicating the variation

TABLE 1

Elliptic cylinder

θ	K	D_L	T_0
0	1·391	0·143	0·391
$\frac{1}{4}\pi$	1·972	0·294	0·972
$\frac{1}{2}\pi$	2·689	0·428	1·689

of K, D_L, and T_0 with θ. The variation of D with L for the three stated angles of incidence is depicted in Fig. 1.

For a circular cylinder with radius equal in length to the semi-major axis of the above elliptic cylinder, i.e. $R = u$, set in the same stream defined by $\beta = \frac{1}{4}\sqrt{3}$, it is found that $K = 2\cdot849$, $D_L = 0\cdot453$, $T_0 = 1\cdot849$, and these values can be compared with those recorded in Table 1.

When the cylinder has the form of a flat plate, then $\xi_0 \to 0$, and $z = 2k\sin\theta$, $\delta = \frac{1}{2}\pi$. Table 2 gives the values of K, D_L, and T_0 for angles

TABLE 2

Flat plate

θ	K	D_L	T_0
0	1·000	0·000	0·000
$\frac{1}{4}\pi$	1·657	0·219	0·657
$\frac{1}{2}\pi$	2·530	0·402	1·530

of incidence $\theta = 0$, $\frac{1}{4}\pi$, $\frac{1}{2}\pi$ assuming that $\beta = \frac{1}{4}\sqrt{3}$, and the length of the plate is $2u$. Then since $h = u$, it follows that $q = \frac{3}{4}$ in this case. In order to indicate the effect of cylinder shape on the displacement of the stagna-

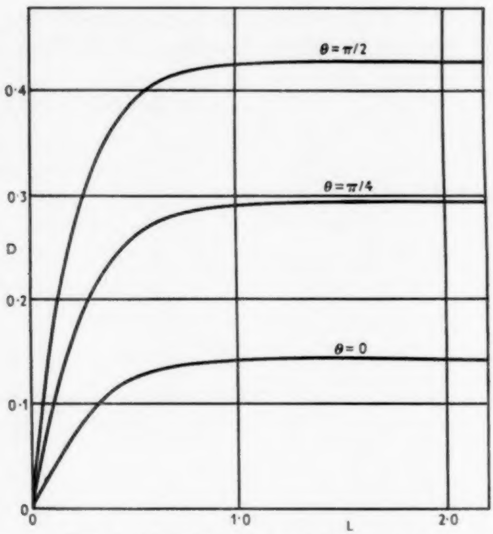

FIG. 1. Elliptic cylinder with $\beta = \frac{1}{4}\sqrt{3}$.

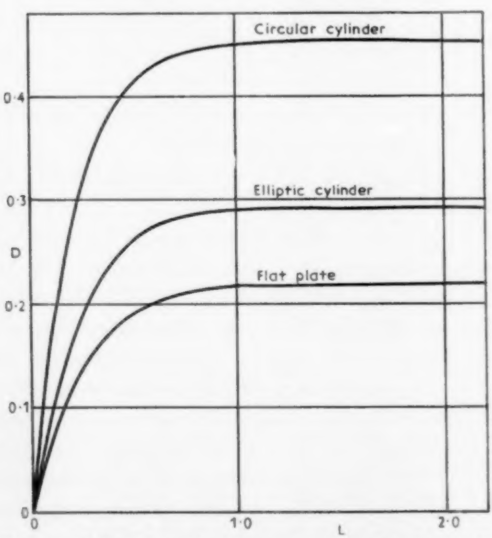

FIG. 2. Cylinders at incidence $\theta = \frac{1}{4}\pi$ with $\beta = \frac{1}{4}\sqrt{3}$.

tion streamline, in Fig. 2 the deflexion parameter D is plotted against the cylinder position parameter L for the flat plate, elliptic cylinder, and circular cylinder at incidence $\frac{1}{4}\pi$.

For streams which are only slightly non-uniform it was shown in section 3 for small q, that $K = 1+O(q)$, hence for such streams from (22) it follows that the corresponding approximation to D becomes

$$D = \beta(K-1)\left\{1-\tfrac{1}{2}(K-1)\tanh^2\frac{L}{\beta}\right\}\tanh\frac{L}{\beta}+O\{(K-1)^3\}. \qquad (23)$$

This result has been investigated for streams with $\beta = \frac{5}{2}\sqrt{3}$ and $\beta = 25\sqrt{3}$, and it is found that change in incidence and cylinder shape follow the

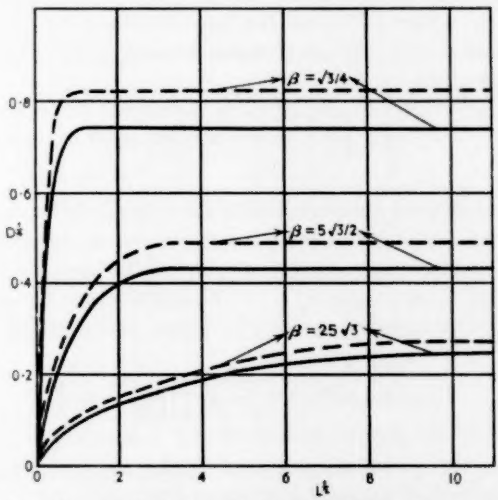

FIG. 3. —— Elliptic cylinder at incidence $\theta = \frac{1}{4}\pi$.
--- Circular cylinder.

same pattern as Figs. 1 and 2. The effect of change in stream form for a given cylinder at a given incidence is exhibited in Fig. 3—the cylinder has the elliptic section defined in conjunction with Fig. 1 and is at an incidence $\theta = \frac{1}{4}\pi$. For comparison the effect of change in stream form on the displacement of the stagnation streamline from a circular cylinder of radius equal to the semi-major axis of the section of the elliptic cylinder has been included—this result is as given by the analysis of Murray and Mitchell (6), with modifications to allow a variation in stream form.

5. The pressure coefficient

The pressure equation for steady rotational flow in the absence of body forces (13, p. 109) is

$$\frac{P}{\rho}+\tfrac{1}{2}v^2-\int\limits_{AB}\nabla^2\psi\,d\psi = C,$$

where C is an absolute constant, B is the point in the fluid at which the pressure P, the density ρ, and the velocity v are measured, and A is any fixed reference point. For the flow with hyperbolic velocity profile it is known that $\nabla^2\psi = \psi/c^2$, and so

$$\frac{P}{\rho}+\tfrac{1}{2}v^2-\frac{1}{2c^2}(\psi^2-\psi_A^2) = C, \tag{24}$$

where ψ is the value of the stream function at B. At large distances from the cylinder, i.e. where $|X| \to \infty$, the flow has the same characteristics as the undisturbed stream. If the pressure at such points is Π, then by use of (3), it follows that

$$\frac{\Pi}{\rho}+\tfrac{1}{2}(1-N^2)V^2+\frac{1}{2c^2}\psi_A^2 = C.$$

The pressure coefficient associated with the cylinder is defined by the ratio $2(P-\Pi)/\rho v_0^2$, where P is the pressure at points on the cylinder surface, and v_0 is the undisturbed stream velocity at the centre of the cylinder, derived from (3), with a value $v_0 = -V(\cosh b+N\sinh b)$, where $b = Y_0/c$. At points on the cylinder $\psi = cpV$, where $p = -K(\sinh b+N\cosh b)$, whence

$$C_p = 1+(K^2-1)\left(\frac{N+\tanh b}{1+N\tanh b}\right)^2-\left(\frac{v}{v_0}\right)^2. \tag{25}$$

By comparison the pressure coefficient C_{p0} for the cylinder in a uniform stream of undisturbed velocity v_0 is given by $1-(v/v_0)^2$.

We now determine v, the velocity in the disturbed stream on the surface of the cylinder. This is the tangential fluid velocity component with a value $H^{-1}\partial\psi/\partial\xi$, for the value $\xi = \xi_0$, and where

$$2H^2 = h^2(\cosh 2\xi_0-\cos 2\eta).$$

By use of (10) it is possible to develop $\partial\psi_0/\partial\xi$, when $\xi = \xi_0$, as a Fourier series with η as the variable, and in this connexion it is found expedient to use equations such as (A 2) of the appendix. Similarly by using the series forms for $\mathrm{ce}_m(\eta, -q)$ and $\mathrm{se}_m(\eta, -q)$ in (9) it is also possible to develop $\partial\Psi/\partial\xi$, when $\xi = \xi_0$, as a Fourier series with η as the variable. In fact

$$\frac{1}{hv_0}\left(\frac{\partial\psi}{\partial\xi}\right)_{\xi=\xi_0} = a_0+\sum_{n=1}^{\infty}(a_n\cos n\eta+b_n\sin n\eta), \tag{26}$$

where the a_n and b_n are known constants depending on the geometry of the stream and the cylinder. It is comparatively easy by use of the above

methods to show for the flat plate in the symmetrical stream that

$$a_{2n} = \frac{1}{k}\tanh b \sum_{m=0}^{\infty} (-1)^{n+m}\{(-1)^m K A_0^{2m} - C_{2m}\} A_{2n}^{2m} F_{2m}(0, -q) \quad (n \geqslant 0),$$

$$a_{2n+1} = \frac{1}{k}\sum_{m=0}^{\infty} (-1)^{n+m+1} C_{2m+1} B_{2n+1}^{2m+1} F_{2m+1}(0, -q) \quad (n \geqslant 0),$$

$$b_{2n} = \frac{2n}{k} I_{2n}(2k\sin\theta)\cot\theta\tanh b \quad (n \geqslant 1),$$

$$b_{2n+1} = \frac{2n+1}{k} I_{2n+1}(2k\sin\theta)\cot\theta \quad (n \geqslant 0).$$

On introducing the constants $A_0 = a_0$, $A_n = \frac{1}{2}(a_n - ib_n)$ for $n \geqslant 1$, it is possible to deduce from (26) that

$$\frac{1}{h^2 v_0^2}\left(\frac{\partial\psi}{\partial\xi}\right)^2_{\xi=\xi_0} = \mu_0 + \sum_{n=1}^{\infty} (\mu_n e^{in\eta} + \bar{\mu}_n e^{-in\eta}), \tag{27}$$

where, for $n \geqslant 0$,

$$\mu_n = \sum_{r=0}^{n} A_r A_{n-r} + 2\sum_{r=1}^{\infty} \bar{A}_r A_{n+r}.$$

The result (27) can thus be used in (25) in order to determine C_p, since

$$\left(\frac{v}{v_0}\right)^2 = \frac{2}{(\cosh 2\xi_0 - \cos 2\eta)}\left(\frac{1}{hv_0}\frac{\partial\psi}{\partial\xi}\right)^2_{\xi=\xi_0}. \tag{28}$$

For the elliptic cylinder in a symmetrical stream it can be shown that

$$C_p = 1 - \frac{e^{2\xi_0}\{1-\cos 2(\theta+\eta)\}}{\cosh 2\xi_0 - \cos 2\eta} + \frac{k\tanh b}{\cosh 2\xi_0 - \cos 2\eta}[e^{-\xi_0}\sin(\theta+\eta) -$$
$$- e^{3\xi_0}\{2\sin(\theta+\eta) - \sin 3(\theta+\eta)\} - e^{\xi_0}\{\sin(\theta+3\eta) + \sin(\theta-\eta)\}] + O(k^2).$$

In order to avoid lengthy expressions this result has been developed only as far as terms of order $O(k)$, and the result can be used to determine the effect of small stream shear on the pressure coefficient. The terms independent of k on the right-hand side of the equation give the value of C_{p0}, the pressure coefficient for the cylinder in a uniform stream (**13**, p. 161). The corresponding result for the flat plate is obtained on substituting $\xi_0 = 0$.

In order to indicate the order of magnitude of the change in pressure coefficient due to non-uniformity in the stream as well as change in incidence, the results for the flat plate in a symmetrical stream of large shear have been investigated—it is assumed that $\beta = \frac{1}{4}\sqrt{3}$, and hence that $q = k^2 = 1$. The coefficients tabulated at various values of the eccentric angle η for the angles of incidence $\theta = \frac{1}{4}\pi$ and $\frac{1}{2}\pi$ are C_{p0}, C_{p1}, and C_{p2}, the pressure coefficients associated with the uniform stream, and the non-

uniform stream when $L = 0{\cdot}25$ and 1 respectively. The values of C_p for larger values of L do not differ appreciably from those of C_{p2}. The infinite pressures at $\eta = 0$, π, and 2π occur as expected, due to the presence of infinite fluid velocities at the trailing and leading edges of the plate.

TABLE 3

Pressure coefficients for the flat plate

η	$\theta = \frac{1}{4}\pi$			$\theta = \frac{1}{2}\pi$		
	C_{p0}	C_{p1}	C_{p2}	C_{p0}	C_{p1}	C_{p2}
0	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$\frac{1}{4}\pi$	-1	$-0{\cdot}92$	$-1{\cdot}33$	0	$-2{\cdot}97$	$1{\cdot}36$
$\frac{1}{2}\pi$	$0{\cdot}5$	$1{\cdot}43$	$2{\cdot}30$	1	$1{\cdot}33$	$2{\cdot}18$
$\frac{3}{4}\pi$	1	$1{\cdot}04$	$1{\cdot}40$	0	$-4{\cdot}45$	$-1{\cdot}43$
π	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$\frac{5}{4}\pi$	-1	$1{\cdot}21$	$2{\cdot}67$	0	$-4{\cdot}45$	$-1{\cdot}43$
$\frac{3}{2}\pi$	$0{\cdot}5$	$0{\cdot}95$	$1{\cdot}39$	1	$1{\cdot}33$	$2{\cdot}18$
$\frac{7}{4}\pi$	1	$1{\cdot}31$	$1{\cdot}90$	0	$-2{\cdot}97$	$1{\cdot}36$
2π	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$

The values for slightly non-uniform streams, when q is small, can similarly be obtained, but the pressure differences in these cases are not so pronounced.

6. The force and moment coefficients

(i) If the lift force on the cylinder in the direction normal to the main stream away from the X-axis is F_Y, and the force in the direction of the negative X-axis is F_X, then

$$F_Y - iF_X = e^{-i\theta} \int\limits_S P\, d\bar{z}.$$

Here θ is the angle of incidence of the cylinder to the main stream, P is the pressure on the cylinder of section contour S, and $\bar{z} = x - iy$, where (x, y) are position coordinates referred to the cylinder axes Ox, Oy. The pressure P is defined by (24), and since ψ is constant on S, then

$$F_Y - iF_X = -\tfrac{1}{2}\rho e^{-i\theta} \int\limits_S v^2\, d\bar{z}.$$

It is now possible to introduce force coefficients defined by

$$C_Y = F_Y/\pi\rho u v_0^2, \qquad C_X = F_X/\pi\rho u v_0^2.$$

From (6) it is seen that $z = h\cosh(\xi_0 + i\eta)$ on S, and $(v/v_0)^2$ is given by (28), hence

$$C_Y - iC_X = \frac{ie^{-i\theta}}{2\pi\cosh\xi_0} \int\limits_0^{2\pi} \left(\frac{1}{hv_0}\frac{\partial\psi}{\partial\xi}\right)^2 \operatorname{cosech}(\xi_0 + i\eta)\, d\eta. \tag{29}$$

But for $\xi_0 > 0$ it is possible to write

$$\operatorname{cosech}(\xi_0+i\eta) = 2\sum_{n=0}^{\infty} e^{-(2n+1)(\xi_0+i\eta)},$$

and with the use of (27), integration of (29) leads to the result

$$C_Y - iC_X = 2ie^{-i\theta}\operatorname{sech}\xi_0 \sum_{n=0}^{\infty} \mu_{2n+1} e^{-(2n+1)\xi_0}. \tag{30}$$

For symmetrical streams with small shear, i.e. with q small, it can be shown for the elliptic cylinder that

$$C_Y = k(e^{2\xi_0} - \cos 2\theta)\operatorname{sech}\xi_0 \tanh b + O(k^2),$$

and the force coefficient C_X is at least of order $O(k^2)$. For the uniform stream it is seen that the force on the cylinder is zero.

For the flat plate in a symmetrical stream higher order approximations can readily be determined under the limiting process $\xi_0 \to 0$, and are recorded here as the expressions involved are not as long as those for the elliptic cylinder. In fact

$$C_Y = 2k\left\{\frac{1}{\lambda_1} + \frac{3k^2}{8}(4\sin^2\theta - 1)\right\}\sin^2\theta \tanh b + O(k^5 \log k),$$

where $$\lambda_1 = 1 + k^2(\log k - \mu - \tfrac{1}{8}) + O(k^4 \log k),$$

and C_X is at least of order $O(k^5 \log k)$.

A numerical analysis of the results of (30) for a flat plate at angles of incidence $\theta = 0$, $\frac{1}{4}\pi$, $\frac{1}{2}\pi$ respectively in symmetrical streams of varying shear defined by $\beta = 25\sqrt{3}$, $\frac{5}{2}\sqrt{3}$, and $\frac{1}{4}\sqrt{3}$ respectively, has been carried out and values of $C_Y/\tanh b$ recorded in Table 4.

TABLE 4

Force coefficient $C_Y/\tanh b$ for flat plate

θ	$\beta = 25\sqrt{3}$	$\frac{5}{2}\sqrt{3}$	$\frac{1}{4}\sqrt{3}$
0	0	0	0
$\frac{1}{4}\pi$	0·012	0·120	1·634
$\frac{1}{2}\pi$	0·023	0·242	18·256

For constant β, i.e. for a given stream, the lift coefficient is proportional to $\tanh(L/\beta)$, the constant of proportionality increasing with increase of angle of incidence in the interval $(0, \frac{1}{2}\pi)$. The relationship $C_Y - L$ for various values of β when the plate is at incidence $\frac{1}{4}\pi$ is exhibited in Fig. 4.

At the other extreme of geometrical shape for the cylinder section, viz. the circular section, by use of the stream function of (6), the lift force can be expressed in the form

$$C_Y = 4\beta \tanh b \sum_{n=1}^{\infty} I_n(\beta^{-1})/K_n(\beta^{-1}),$$

where $\beta = c/R$. For the three stream forms of Table 4 the corresponding results for $C_Y/\tanh b$ are 0·046, 0·496, and 51·904. The force coefficient C_X is zero in magnitude.

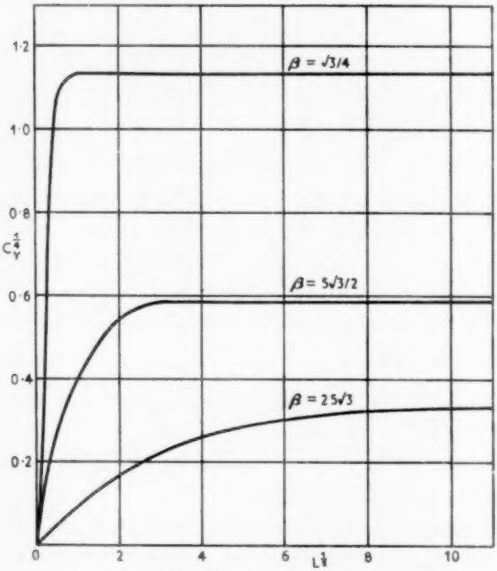

FIG. 4. Flat plate at incidence $\theta = \frac{1}{4}\pi$.

(ii) The couple acting on the cylinder in an anticlockwise direction has a moment M about the axis of the cylinder through O given by

$$M = \mathrm{re} \int_S Pz\, d\bar{z}.$$

By use of (24), and the result

$$\mathrm{re} \int_S z\, d\bar{z} = \tfrac{1}{2} \int_S d(z\bar{z}) = 0,$$

it follows that the moment of the couple reduces to

$$M = \mathrm{re}\left(-\tfrac{1}{2}\rho \int_S v^2 z\, d\bar{z}\right).$$

If the moment coefficient is defined as $C_M = M/\pi\rho u^2 v_0^2$, then as in (i) of this section

$$C_M = \mathrm{re}\frac{i}{2\pi}\operatorname{sech}^2\xi_0 \int_0^{2\pi} \left(\frac{1}{hv_0}\frac{\partial\psi}{\partial\xi}\right)^2 \coth(\xi_0+i\eta)\, d\eta. \tag{31}$$

But for $\xi_0 > 0$, it is known that

$$\coth(\xi_0+i\eta) = 1+2\sum_{n=1}^{\infty} e^{-2n(\xi_0+i\eta)},$$

hence on using (27), and integrating (31), there results

$$C_M = \operatorname{re} 2i\operatorname{sech}^2\xi_0 \sum_{n=1}^{\infty} \mu_{2n} e^{-2n\xi_0}, \tag{32}$$

it being known that μ_0 is a real constant.

For the elliptic cylinder in a symmetrical stream of small shear it can be shown that (32) reduces to the result (**13**, p. 165),

$$C_M = \tfrac{1}{2}\sin 2\theta \operatorname{sech}^2\xi_0 + O(k^2),$$

and for the flat plate, allowing $\xi_0 \to 0$,

$$C_M = \frac{1}{2}\left\{\frac{1}{\lambda_2}\left[1+\frac{k^2}{8}(16\sin^2\theta-3)\right]+\frac{k^4}{384}[37-352\sin^2\theta+576\sin^4\theta]\right\}\sin 2\theta+$$
$$+\tfrac{1}{2}\{k^2+\tfrac{1}{3}k^4(3\sin^2\theta+1)\}\sin^2\theta\sin 2\theta\tanh^2 b+O(k^6),$$

where

$$\lambda_2 = 1+k^2(\log k-\mu-\tfrac{1}{8})+\tfrac{3}{8}k^4(\log k-\mu-\tfrac{79}{144})+O(k^6\log k).$$

A similar type of numerical analysis as applied in (i) of this section can also be applied here. It is found that for angles of incidence $\theta = 0$, $\frac{1}{2}\pi$ the moment coefficient is zero in magnitude, and for $\theta = \frac{1}{4}\pi$ the values of C_M are given by Table 5.

TABLE 5

Moment coefficient C_M for flat plate

β	C_M
$25\sqrt{3}$	$0{\cdot}500+0{\cdot}000\tanh^2 b$
$\frac{3}{4}\sqrt{3}$	$0{\cdot}521-0{\cdot}003\tanh^2 b$
$\frac{1}{4}\sqrt{3}$	$0{\cdot}440+0{\cdot}786\tanh^2 b$

The action of the couple is to orientate the plate to a position transverse to the main stream. The couple associated with a circular cylinder has zero moment about its axis.

7. Conclusions

Figs. 1, 2, and 3 indicate that as the cylinder is stationed further from the axis of symmetry of the stream, the non-dimensional displacement of the stagnation streamline increases from zero to a maximum limiting value, the latter being reached at critical distances which are approxi-

mately independent of incidence and shape of cylinder section but increase the more uniform the stream. For a fixed position of the cylinder the displacement increases (i) with incidence, the rate of increase being greater at smaller angles of incidence in the range $0 \leqslant \theta \leqslant \frac{1}{2}\pi$, (ii) the greater the transverse dimensions of the cylinder section, (iii) the more non-uniform the stream. In all cases the displacement of the stagnation streamline is towards a region of higher stream velocity.

Table 3 illustrates how a change in stream form and change in incidence affects the pressure coefficient at various points on the flat plate type of cylinder. The effects are much more pronounced when the non-uniformity of the stream is more pronounced, i.e. for streams of greater shear, and the pressure distribution becomes asymmetric with respect to the centre point of the straight line section of the plate.

Although by virtue of the complexity of the equations involved in the analysis it has not been found possible to prove that the force on the cylinder in the direction of the main stream is zero, it is conjectured that this is so in view of the results for the elliptic cylinder in a stream with small shear, and the numerical results for a flat plate in various streams at various angles of incidence.

When the cylinder is at incidence $\theta = \frac{1}{2}\pi$ to the main stream, it is acted on by a force in the direction of the major axis of its right section. A similar state of affairs exists when the stream has constant vorticity (**2**), and as noted by James (**2**) it is evident that as regards this transverse force effect the shear flow with distributed vorticity is analogous to the potential flow with circulation round the cylinder.

A comparison of Figs. 3 and 4, and Table 5, shows that the lift force and moment coefficients exhibit the same general characteristics as the displacement of the stagnation streamline. The lift force coefficient for a symmetrical stream is directly proportional to $\tanh b$, where $b = Y_0/c$, and has a maximum limiting value which is greater for larger angles of incidence in the range $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. It is noticed from Fig. 4 that the effect of an increase in the non-uniformity of the stream is to increase the magnitude of the lift coefficient. By comparison of the lift force on the flat plate and the circular cylinder, it is seen that the effect of increase of length of the semi-minor axis of the cylinder section is to increase the magnitude of the lift force at given incidence and position of the cylinder.

The moment coefficient is a linear function of $\tanh^2 b$, and this also has a maximum limiting value. This coefficient is zero however at the extremes of the range $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. There is an increase in the magnitude of the moment coefficient as the stream becomes more non-uniform.

APPENDIX

In order to evaluate the integral $C_{2n}(k)$ of (12) subsidiary variables z, δ are introduced such that

$$2k\cosh\xi_0\sin\theta = z\sin\delta, \qquad 2k\sinh\xi_0\cos\theta = z\cos\delta.$$

Hence

$$\left.\begin{aligned}\tan\delta &= \coth\xi_0\tan\theta \\ z^2 &= 2q(\cosh 2\xi_0-\cos 2\theta)\end{aligned}\right\}, \tag{A 1}$$

and thus $2kw_0 = z\sin(\eta+\delta)$ in $C_{2n}(k)$. But it is known that

$$e^{z\sin(\eta+\delta)} = I_0(z)+2\sum_{r=1}^{\infty}(-1)^r I_{2r}(z)\cos 2r(\eta+\delta)+$$

$$+2\sum_{r=0}^{\infty}(-1)^r I_{2r+1}(z)\sin(2r+1)(\eta+\delta), \tag{A 2}$$

hence on substituting the series representation for $\mathrm{ce}_{2n}(\eta,-q)$,

$$C_{2n}(k) = \frac{1}{2\pi}\sum_{r=0}^{\infty}(-1)^{n+r}A_{2r}^{2n}\int_0^{2\pi} e^{z\sin(\eta+\delta)}\cos 2r\eta\,d\eta$$

$$= (-1)^n\sum_{r=0}^{\infty}A_{2r}^{2n}I_{2r}(z)\cos 2r\delta, \tag{A 3}$$

where z and δ are as defined in (A 1).

If k changes sign then from (A 1) the angle δ is replaced by $\pi+\delta$ in the above result, hence $C_{2n}(k) = C_{2n}(-k)$.

In a similar manner it can be shown that

$$C_{2n+1}(k) = (-1)^n\sum_{r=0}^{\infty}B_{2r+1}^{2n+1}I_{2r+1}(z)\sin(2r+1)\delta, \tag{A 4}$$

$$S_{2n+1}(k) = (-1)^n\sum_{r=0}^{\infty}A_{2r+1}^{2n+1}I_{2r+1}(z)\cos(2r+1)\delta, \tag{A 5}$$

$$S_{2n+2}(k) = (-1)^n\sum_{r=0}^{\infty}B_{2r+2}^{2n+2}I_{2r+2}(z)\sin(2r+2)\delta, \tag{A 6}$$

the results for change in sign of k following immediately.

Following an integral equation approach (**10**, p. 202) alternative expressions for (A 3)–(A 6) are determined in the form

$$\begin{aligned}p_{2n}C_{2n}(k) &= \mathrm{Ce}_{2n}(\xi_0,-q)\mathrm{ce}_{2n}(\theta,q),\\ p_{2n+1}S_{2n+1}(k) &= \mathrm{Se}_{2n+1}(\xi_0,-q)\mathrm{ce}_{2n+1}(\theta,q),\\ s_{2n+1}C_{2n+1}(k) &= \mathrm{Ce}_{2n+1}(\xi_0,-q)\mathrm{se}_{2n+1}(\theta,q),\\ s_{2n+2}S_{2n+2}(k) &= -\mathrm{Se}_{2n+2}(\xi_0,-q)\mathrm{se}_{2n+2}(\theta,q),\end{aligned}$$

where the functions appearing in these results are defined in (**10**).

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ON THE CALCULATION OF EDDY VISCOSITY AND HEAT TRANSFER IN A TURBULENT BOUNDARY LAYER NEAR A RAPIDLY ROTATING DISK

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SUMMARY

In this paper the distribution of the radial component of Reynolds shearing stress is calculated in the turbulent boundary layer over a rapidly rotating disk, by an integration of the equations of *mean* flow, the validity of the von Kármán velocity profiles being assumed. The distribution of eddy heat diffusivity is then evaluated by applying Reynolds analogy in a thin region of flow very near the surface of the disk, and using the approximate similarity of the radial component of disk flow and flat plate flow to extend the results through the remainder of the inner, crucial, part of the boundary layer (about 16 per cent of the layer thickness). The method of calculation developed by Davies and Bourne (**5**) is then applied to evaluate, in the case of air, the rate of heat transfer from the disk, when the surface of the plate is kept at a constant temperature. It is assumed that the disk is rotating about its centre sufficiently rapidly to neglect (*a*) the completely laminar zone of flow near the centre, and (*b*) the thickness of the laminar sublayer; under these conditions the calculated value of heat transfer is found to be in good agreement with the value suggested by the experiments of Cobb and Saunders (**4**).

1. Introduction

THE flow due to a disk rotating in its own plane was considered first by von Kármán (**1**). He considered both the laminar and the turbulent cases and obtained, by his integral method, expressions for the velocity components in the boundary layer flow over the disk. The distributions of circumferential and radial velocity components have been measured experimentally by Gregory, Stuart, and Walker (**2**), who found agreement with von Kármán's distributions when the flow was laminar. Good agreement was obtained in the turbulent conditions for the circumferential velocity, and reasonably good agreement was also obtained for the radial component in the inner part of the boundary layer.

In the laminar case the associated problem of calculating the rate of heat transfer has been solved by Millsaps and Pohlhausen (**3**), when the surface temperature of the disk is independent of position. No solution has been given, however, in the important case of turbulent flow, which occurs in most practical applications.

Experimental results involving heat transfer into air have been obtained by Cobb and Saunders (**4**), when the flow over the disk was in part laminar and in part turbulent. It was shown that, as the Reynolds number in-

creases, the transition point from laminar to turbulent flow moves inwards and an increasing proportion of the disk surface comes under turbulent conditions. Cobb and Saunders suggested that in the limiting case of a very rapidly spinning disk, when the completely laminar zone near the centre of the disk can be neglected, the experimental results approach values given by the expression

$$N = 0{\cdot}015R^{0{\cdot}8},$$

where N and R are appropriate Nusselt and Reynolds numbers for disk flow; here $R = \omega r^2/\nu$, r and ω denoting the radius and angular velocity of the disk, ν the kinematic viscosity of the ambient fluid;

$$N = Hr/k(T_1 - T_0),$$

k denoting the thermal conductivity of the fluid, T_1 and T_0 the temperature of the disk and the ambient fluid, and H the average rate of heat transfer per unit area of the disk.

In order to formulate a corresponding theoretical approach the analysis developed by Davies and Bourne (**5**), for heat transfer from a flat plate, is applied in this paper to the problem of a rapidly rotating disk. It is assumed that the disk is rotating sufficiently rapidly to neglect (*a*) the completely laminar zone of flow near the centre, and (*b*) the thickness of the laminar sublayer, these being the limiting conditions in which Kármán's velocity profiles apply. Reynolds stresses are then considerably greater than viscous stresses over the whole flow field.

The equation involving the Reynolds stress component, associated with radial and normal eddy velocities, is first integrated, following the usual boundary layer approximations and applying the expressions given by von Kármán for the radial and circumferential velocity components. The experimental results of Gregory, Stuart, and Walker (**2**) show that the position of the maximum radial component of velocity is given accurately by Kármán's expression and fairly good agreement is obtained in the inner part of the boundary layer. A considerable degree of error is found in the outer part of the layer. It should be noted, however, that a negligible completely laminar zone is assumed by von Kármán, whereas the Reynolds number involved in the measurements is not sufficiently high to permit the completely laminar zone at the centre to be neglected. We note that near the surface von Kármán assumes a one-seventh power law profile for the *radial* component of flow, because this is approximately valid in a turbulent boundary layer flow over a flat plate, and it is reasonable to suppose that the two flows are similar in the inner part of the boundary layers. The ensuing calculated values of Reynolds stress are expected to be accurate only in the inner part of the boundary layer

(about 16 per cent of the layer thickness), but, as shown by Davies and Bourne (**5**), this is the crucial region for heat transfer calculations.

Using the concept of Reynolds analogy, which has been shown to be valid in forced convection from a flat plate, the distribution of eddy heat diffusivity, ϵ_H, is then evaluated in the region of flow which is immediately adjacent to the surface of the disk (about 2 per cent of the thickness of the boundary layer). A power law representation of ϵ_H in this inner flow is then constructed with good accuracy, and the dependence of this power law on distance from the surface of the disk is found to be in agreement with that given by a corresponding flat plate flow. Reynolds analogy is shown to break down at points further distant from the disk. However, since the von Kármán profile of radial velocity for the disk flow is in close agreement with the flat plate profile in the inner 16 per cent or so of the boundary layer thickness, the power law for ϵ_H is assumed to be valid throughout this part (the inner 16 per cent) of the layer.

The method of sources is next applied, the heated disk being regarded as an assembly of concentric circular sources. The radial distribution of source strength (i.e. of local rate of heat transfer) is determined by solving an integral equation, the temperature of the disk being taken to be constant over its surface.

The calculated results are found to satisfy the relation

$$N = 0{\cdot}014R^{0{\cdot}8}.$$

This is in good agreement with the result suggested by Cobb and Saunders (**4**).

2. The equations of fully turbulent mean flow and of mean temperature over a rotating disk

In order to set up the Reynolds equations of mean flow in the fully turbulent case we neglect the viscosity terms in the laminar flow equations and, applying the usual Reynolds averaging technique, we obtain the equations of mean motion (see, for example, Goldstein, **6**). When expressed in cylindrical coordinates r (the distance from the centre of the disk) and z (the normal distance from the disk surface), these are

$$U\frac{\partial U}{\partial r}+W\frac{\partial U}{\partial z}-\frac{V^2}{r} = -\frac{1}{r}\frac{\partial}{\partial r}[r(\overline{u'u'})]-\frac{\partial}{\partial z}(\overline{u'w'})+\frac{(\overline{v'v'})}{r}, \tag{1}$$

$$U\frac{\partial V}{\partial r}+W\frac{\partial V}{\partial z}+\frac{UV}{r} = -\frac{1}{r}\frac{\partial}{\partial r}[r(\overline{u'v'})]-\frac{\partial}{\partial z}(\overline{v'w'})-\frac{(\overline{u'v'})}{r}, \tag{2}$$

and

$$U\frac{\partial W}{\partial r}+W\frac{\partial W}{\partial z} = -\frac{\partial}{\partial r}(\overline{u'w'})-\frac{\partial}{\partial z}(\overline{w'w'})-\frac{(\overline{u'w'})}{r}; \tag{3}$$

U, V, and W denote mean velocities in the radial, circumferential, and normal directions respectively; u', v', w' denote the corresponding components of eddy velocity; $\rho(\overline{u'u'})$, $\rho(\overline{v'v'})$, $\rho(\overline{w'w'})$, $\rho(\overline{u'w'})$, $\rho(\overline{u'v'})$, $\rho(\overline{v'w'})$ denote the components of Reynolds stress, where ρ is the fluid density. The equation of continuity of the mean flow is

$$\frac{1}{r}\frac{\partial}{\partial r}(rU)+\frac{\partial W}{\partial z} = 0, \tag{4}$$

and the equation of mean temperature (neglecting friction heating and molecular heat transfer) is

$$U\frac{\partial T}{\partial r}+W\frac{\partial T}{\partial z} = -\frac{\partial}{\partial r}(\overline{u'T'})-\frac{\partial}{\partial z}(\overline{w'T'})+\frac{(\overline{u'T'})}{r}, \tag{5}$$

where T' denotes the departure from the mean temperature, $\rho c_p(\overline{u'T'})$, $\rho c_p(\overline{w'T'})$ denote mean flux of heat in the radial and normal directions, and c_p is the specific heat of the fluid at constant pressure. In a turbulent boundary layer on a flat plate it is known from experiment (see, for example, Townsend, **7**) that the various components of Reynolds stress are of the same order of magnitude, and, since the flow in the inner part of the boundary layer over the disk is similar to the flow over a flat plate, this result is probably valid in the inner part of disk flow. If we then make the usual boundary layer approximation and retain only terms on the right-hand sides of equations (1), (2), (3), and (5) involving derivatives with respect to z, we obtain equations involving only the three components of Reynolds stress associated with w'.

The reduced forms of (1) and (5) only are required in the present calculation, and we obtain

$$U\frac{\partial U}{\partial r}+W\frac{\partial U}{\partial z}-\frac{V^2}{r} = -\frac{\partial}{\partial z}(\overline{u'w'}), \tag{6}$$

and

$$U\frac{\partial T}{\partial r}+W\frac{\partial T}{\partial z} = -\frac{\partial}{\partial z}(\overline{w'T'}). \tag{7}$$

In equation (6), U and V are given by the von Kármán profiles (**1**), viz.

$$U = \alpha\omega r\xi^{1/7}(1-\xi), \tag{8}$$

and

$$V = \omega r(1-\xi^{1/7}), \tag{9}$$

where $\alpha = 0{\cdot}526$ and $\xi = z/\delta$, δ being the thickness of the boundary layer, as defined in Kármán's integral method of calculation, and given by

$$\delta = \alpha(\nu/\omega)^{1/5}r^{3/5}. \tag{10}$$

An expression for the vertical mean velocity component is now obtained from (4), which can be written in the form

$$\frac{1}{\delta}\frac{\partial W}{\partial \xi} = \frac{\partial W}{\partial z} = -\frac{U}{r} - \frac{\partial U}{\partial r},$$

where the right-hand side is a function of ξ only. By integration we obtain the form

$$W = \alpha\omega\delta\left(-\frac{67}{40}\xi^{8/7} + \frac{46}{75}\xi^{15/7}\right). \tag{11}$$

3. Calculation of the distribution of radial component of Reynolds shearing stress over a rotating disk

Substitution of (8), (9), and (11) into equation (6) for the radial component of Reynolds stress, $\rho(\overline{u'w'})$, and integration yields

$$(\overline{u'w'}) = -\frac{\tau_r}{\rho} - \alpha^3\omega^{9/5}\nu^{1/5}r^{8/5}\left\{-\frac{1}{\alpha^2}\xi + \frac{7}{4\alpha^2}\xi^{8/7} + \right.$$

$$\left. + \frac{7}{9}\left(\frac{189}{280} - \frac{1}{\alpha^2}\right)\xi^{9/7} + \frac{203}{600}\xi^{16/7} - \frac{203}{1725}\xi^{23/7}\right\}, \tag{12}$$

where τ_r is the radial component of shearing stress at the surface, and is derived by von Kármán's method (**1**) to be

$$\tau_r = 0{\cdot}0225\rho(\alpha r\omega)^{7/4}\nu^{1/4}\delta^{-1/4}(1+\alpha^{-2})^{3/8}. \tag{13}$$

The calculated values of $(\overline{u'w'})$ in the inner boundary layer are shown in Fig. 1, demonstrating a change in sign at $\xi = 0{\cdot}042$, from the zone dominated by the constraining effect of the surface of the disk in the radial direction to the zone dominated by the centrifugal effect.

4. Calculation of heat transfer from a rapidly rotating disk

In order to obtain a solution of the associated heat transfer problem, the Reynolds analogy is now assumed to be valid in the region of flow immediately adjacent to the surface of the disk. We express this mathematically by the relations

$$\frac{(\overline{u'w'})}{\partial U/\partial z} = -\epsilon, \qquad \frac{(\overline{T'w'})}{\partial T/\partial z} = -\epsilon_H,$$

and

$$\epsilon = \epsilon_H, \tag{14}$$

where ϵ and ϵ_H are respectively the eddy viscosity and eddy heat diffusivity. Using (14), the mean temperature equation (7) becomes

$$U\frac{\partial T}{\partial r} + W\frac{\partial T}{\partial z} = \frac{\partial}{\partial z}\left(\epsilon\frac{\partial T}{\partial z}\right). \tag{15}$$

The left-hand side of equation (15) is then simplified by an appropriate

von Mises transform (**8**, p. 126), so that r and z are replaced by r and ψ (the stream function of the mean flow) as independent variables. We obtain

$$\frac{\partial T}{\partial r} = r^2 \frac{\partial}{\partial \psi}\left(\epsilon U \frac{\partial T}{\partial \psi}\right). \tag{16}$$

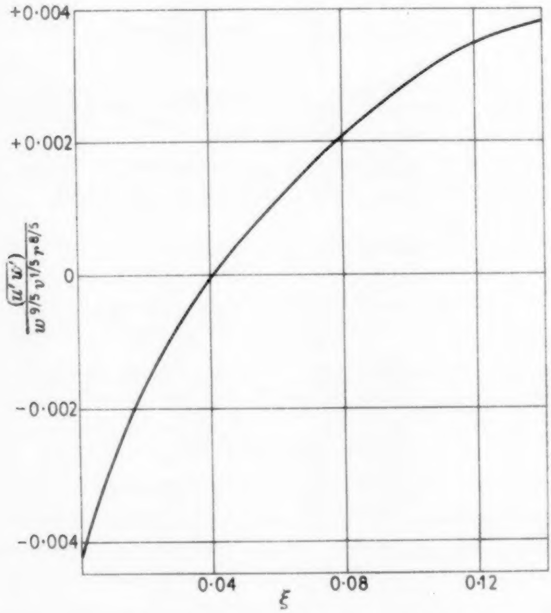

FIG. 1. Distribution of radial component of Reynolds stress, $\rho(\overline{u'w'})$, near a very rapidly spinning disk.

In order to apply the convenient method of calculating heat transfer employed by Davies and Bourne (**5**), the term ϵU must now be expressed in terms of ψ. An expression for ψ in terms of r and ξ is first obtained by integrating the relations

$$\frac{\partial \psi}{\partial z} = rU \quad \text{and} \quad \frac{\partial \psi}{\partial r} = -rW, \tag{17}$$

which replace the continuity equation. Using the von Kármán form (**8**) for U, this integration yields

$$\psi = \alpha^2 \omega^{4/5} \nu^{1/5} r^{13/5} (\tfrac{7}{8}\xi^{8/7} - \tfrac{7}{15}\xi^{15/7}). \tag{18}$$

An expression for ϵU in terms of r and ξ is then obtained from (8), (12),

(13), and (14): this is

$$\epsilon U = 7\alpha^2\omega^{8/5}\nu^{2/5}\xi(1-\xi)(1-8\xi)^{-1}r^{11/5}\Big[0{\cdot}0225\alpha^{-1/4}(1+\alpha^2)^{3/8}+$$

$$+\alpha\left\{-\frac{1}{\alpha^2}\xi+\frac{7}{4\alpha^2}\xi^{8/7}+\frac{7}{9}\left(\frac{189}{280}-\frac{1}{\alpha^2}\right)\xi^{9/7}+\frac{203}{600}\xi^{16/7}-\frac{203}{1725}\xi^{23/7}\right\}\Big]. \quad (19)$$

The variation of $\psi/r^{13/5}$ with $\epsilon U/r^{11/5}$, as given by (18) and (19), is shown in Fig. 2 up to $\xi = 0{\cdot}04$.

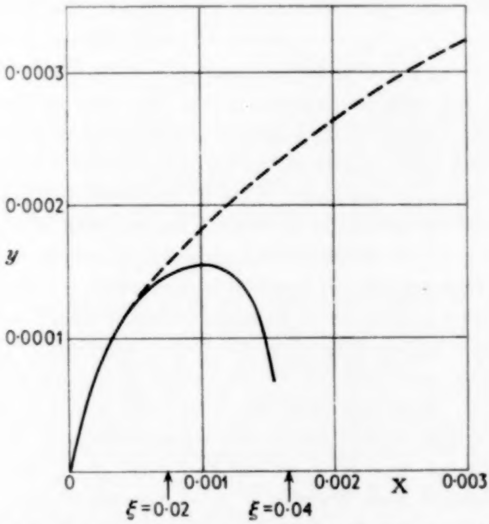

FIG. 2. Variation of $\epsilon_H U$ with ψ; $y = \epsilon_H U/(\omega^{8/5}\nu^{2/5}r^{11/5})$, $X = \psi/(\omega^{4/5}\nu^{1/5}r^{13/5})$; —— calculated from Reynolds analogy, – – – $y = 0{\cdot}0075X^{0{\cdot}54}$.

We note at this stage that all the available experimental evidence concerning forced convection in a turbulent boundary layer on a flat plate supports the Reynolds analogy method of linking through relation (14), momentum and heat transfer. It is found in the case of the disk flow that the calculated distribution of Reynolds stress in the thin region $\xi < 0{\cdot}02$ leads to a distribution of ϵ which is similar to that in flat plate flow. Consequently we have taken ϵ and ϵ_H to be identical for disk flow in this region. However, at $\xi = 0{\cdot}042$, the calculated value of ϵ is already zero, and the associated zero value then given by Reynolds analogy for ϵ_H is *not physically reasonable*. There must be a mean, outward, *non-zero* flux of heat from the disk throughout the boundary layer together with a finite gradient of temperature at all points in the layer. Consequently relation (14) cannot be satisfied with $\epsilon_H = 0$, and Reynolds analogy breaks down

in this neighbourhood. Shear flow turbulence theory, in its present stage of development, cannot be used to predict the form of ϵ_H over the whole thickness of the boundary layer, and we make the *assumption* that the values of $U\epsilon_H$ calculated in the thin layer immediately adjacent to the disk surface can be extrapolated smoothly, through the part of the boundary layer which is likely to influence strongly the rate of heat transfer from the surface. This can be justified in the following way.

We first suppose that $\epsilon_H U$ and ψ can be related by a good power law representation in the inner part of the boundary layer $\xi < 0{\cdot}02$. We write

$$y = bX^t, \tag{20}$$

where $\quad y = \epsilon_H U/(\omega^{8/5}\nu^{2/5}r^{11/5}) \quad$ and $\quad X = \psi/(\omega^{4/5}\nu^{1/5}r^{13/5})$.

The variation of y with X is shown in Fig. 2: a very good representation, with $b = 0{\cdot}0075$ and $t = 0{\cdot}54$ is found possible with 4 per cent deviation at most from the exact values in the region $\xi < 0{\cdot}02$. As in the flat plate case (**5**), the ensuing calculated rates of heat transfer are found not to depend critically on the choice of b and t; i.e. any pair of values of b and t which yields a good fit to the actual variation of y with X leads to calculated rates of heat transfer in reasonable agreement with each other. The particular value $t = 0{\cdot}54$ (with an appropriate b value) is taken, since it is also suggested by the following alternative approach.

In considering the extension of the values of ϵ_H beyond $\xi = 0{\cdot}02$, we note that in the case of turbulent flow over a flat plate Davies and Bourne have shown (**5**) that a good power law representation of the ϵ_H variation in the inner 16 per cent or so of the boundary layer thickness leads to calculated values of heat transfer in good agreement with experimental values. We infer that the flow properties in the outer region of the boundary layer do not influence significantly the rate of heat transfer at the surface; this is due to the fact that the mean velocity increases very sharply in the part of a turbulent boundary layer near the surface and consequently the main flux of heat takes place in this region. It is now reasonable to suppose that a similar result applies in the case of a rotating disk. In this problem we find that a one-seventh power law velocity profile constitutes a good approximation to the von Kármán profile for radial flow in the inner 16 per cent or so of the boundary layer. It can be shown by calculation that the factor $(1-\xi)$ in equation (8) can be effectively neglected in this part of the layer and (8) can be written in the form

$$U/U_0 = \xi^{1/7} = [z/\{\alpha(\nu/\omega)^{1/5}r^{3/5}\}]^{1/7} \tag{21}$$

where $U_0 = \alpha\omega r$, so that $U/U_0 = 1$ at $\xi = 1$. Comparing now (21) with the flat plate profile discussed by Davies and Bourne (**5**)

$$U/U_0 = \eta^{1/7},$$

where $\eta = y/(kx^q)$ and x and y are downstream and normal coordinates, we see that the profile parameter denoted by q in their work (**5**, equation (3)) has the corresponding value $\frac{3}{5}$ in the disk problem. We then *assume* that the distributions of ϵ_H in the radial component of disk flow and flat plate flow are similar in form: this is physically reasonable, because we would expect ϵ_H *in general* to depend on the mean distribution of turbulent energy (independently of any changes in sign of Reynolds stress), and this in turn to depend on the distribution of mean velocity profile. The velocity profiles are practically identical in form in the inner part of the layer, and so we write for the disk flow

$$\epsilon_H/(KU_0 r^p) = \xi^{0\cdot 46}, \tag{22}$$

where the parameter $p = 2q-1 = \frac{1}{5}$ and K is a constant to be determined, since this form (22) is known to give a good representation of the ϵ_H distribution in the inner part of the layer in the flat plate case (**5**, equation (2)). We infer, by using (21) and (22), that in this region of the disk flow

$$\epsilon_H U/(K\chi^2\omega^2 r^{11/5}) = \xi^{0\cdot 61}. \tag{23}$$

Equation (18) can be used to show that in the inner 16 per cent of the boundary layer ψ is proportional to $\xi^{8/7}$ to a high degree of approximation and, using (23), $\epsilon_H U$ is seen to be proportional to $\psi^{0\cdot 54}$, which is in agreement with the result given by (20) using $t = 0{\cdot}54$. The range of validity of (23) includes the region $\xi < 0{\cdot}02$, where the appropriate value of the parameter b is known, and this enables the parameter K to be evaluated. The form $y = 0{\cdot}0075X^{0\cdot 54}$ can then be applied over the inner 16 per cent or so of the boundary layer thickness.

We note that a theory developed on these premisses cannot be expected to predict accurately the distribution of temperature in the outer regions of the boundary layer.

Substitution now of (20) into the temperature equation (16) leads to the equation

$$\frac{\partial T}{\partial R} = c\frac{\partial}{\partial \psi}\left(\psi^t \frac{\partial T}{\partial \psi}\right), \tag{24}$$

where $R = r^{13(2-t)/5}$ and $c = 5b\omega^{4(2-t)/5}\nu^{(2-t)/5}/(26-13t)$. Equation (24) is identical in form with the basic equation discussed previously by Davies and Bourne (**5**, equation (11)). We suppose that a continuous uniform circular source, radius r_0, is situated on the surface of the disk, is concentric with the disk, and emits Q units of heat per unit time, per unit length of source. The analysis of the flat plate solution (**5**) can then be applied and a solution giving the distribution of local rate of heat transfer Q obtained for any prescribed radial distribution of surface temperature. In the

particular case of *constant* disk temperature, the distribution of $Q(r_0)$ is given by

$$Q = K_1 r_0^{3/5}, \tag{25}$$

where

$$K_1 = (13/5)^{(1-t)/(2-t)}(2-t)^{1/(2-t)}\Gamma\{1/(2-t)\}\sin\{\pi/(2-t)\}\times$$
$$\times \rho c_p(T_1-T_0)b^{1/(2-t)}\omega^{4/5}\nu^{1/5}.$$

The total heat flux from a circular area of the disk of radius r is given by

$$E = \int_0^r 2\pi r_0\, Q(r_0)\, dr_0 = \tfrac{10}{13}\pi K_1 r^{3/5}, \tag{26}$$

the average rate of heat transfer over a circular area of radius r by

$$H = E/(\pi r^2) = \tfrac{10}{13}K_1 r^{3/5},$$

and the Nusselt number by

$$N = rH/[k(T_1-T_0)] = K_2\sigma R^{4/5}, \tag{27}$$

where

$$K_2 = 2(5/13)^{1/(2-t)}(2-t)^{1/(2-t)}\Gamma\{1/(2-t)\}\sin\{\pi/(2-t)\}b^{1/(2-t)},$$

and σ is the Prandtl number.

The dependence of N on σ and on R is in agreement with that given by Cobb and Saunders from measured values; it is independent of the choice of b and t. The calculated value of σK_2, using $b = 0{\cdot}0075$ and $t = 0{\cdot}54$, is found to be 0·014, which is in good agreement with the value 0·015 suggested by Cobb and Saunders (**4**) for the limiting case of *all turbulent* flow (Cobb and Saunders note that the accuracy to which the coefficient 0·015 is known is probably 5 per cent).

At smaller angular velocities this coefficient (σK_2) will decrease due to the presence of an appreciable zone of completely laminar flow near the centre of the disk and also due to the presence of a significant laminar sublayer. An extension of the analysis is needed to include these factors, but the problem is complicated by the fact that the sub-layer thickness decreases with increasing distance from the centre of the disk (this follows from the definition of the sublayer thickness $y_* = 30\rho/\nu\tau_r$, since τ_r increases with increasing r): the sublayer correction used in the flat plate case (**5**) is then difficult to apply.

Finally, it is suggested that this method of calculating heat transfer in a turbulent boundary layer over a rotating disk and over a flat plate (previously discussed, **5**) is probably capable of extension to other important problems, such as those involving a turbulent boundary layer on

a rotating heated sphere, although the boundary layer approximation is unlikely to be valid in the immediate vicinity of the equatorial plane.

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HEAT FLOW TOWARDS A MOVING CAVITY

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SUMMARY

A problem arising out of the study of heat flow in deep level gold mines is that of the heat flow towards a stope from the surrounding rock.

In Part I a stope is treated as an infinitely long cylindrical cavity of circular cross-section moving with constant velocity U, in a medium of constant uniform thermal conductivity K. The surface of the stope is at a uniform temperature V_0, while at points infinitely distant from the axis of the stope the temperature is equal to virgin rock temperature V_R, the vertical temperature gradient being neglected. A formula for the amount of heat entering unit length of the stope in the steady state per unit time is derived.

In Part II the problem is extended to the case where the cross-section of the stope is elliptical.

PART I

1. Statement of the problem

In a mine, the general scheme of stoping operations is to blast away rock from the stope face periodically, ore being sent to the surface. For purposes of ventilation, a wall, constructed from the remainder of the rock not sent to the surface, is built parallel to the stope face in the case of a parallel wall stope. In the present paper the impulsive velocity of progress is 'smoothed out' to one of continuous motion.

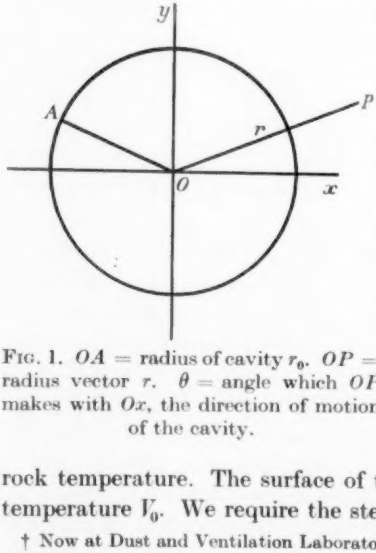

FIG. 1. OA = radius of cavity r_0. OP = radius vector r. θ = angle which OP makes with Ox, the direction of motion of the cavity.

In this Part we will consider the problem of an infinitely long cylindrical stope of circular cross-sectional area, moving with uniform velocity U in a direction perpendicular to the generator of the stope, in an infinite medium of constant uniform thermal conductivity K, the temperature infinitely distant from the axis of the stope being equal to V_R, the virgin rock temperature. The surface of the stope is maintained at a constant temperature V_0. We require the steady state temperature distribution in

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the surrounding rock. Since the stope considered is infinitely long, the problem can be considered as two-dimensional, specified by the coordinates of Fig. 1.

2. Solution

The equation of conduction of heat to be solved is, according to Wilson ((**1**) 408):

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{U}{D}\frac{\partial V}{\partial x}=0, \tag{1}$$

where D is the diffusivity of the medium, V is the temperature in the medium and U is the velocity of the cavity. (U is replaced by $-U$ in the formulae quoted in the above paper.) This corresponds to a motion of the medium in the negative x-direction. The boundary conditions to be satisfied are:

$$\begin{aligned} V &= V_0 \quad \text{on the surface of the cavity } x^2+y^2 = r_0^2, \\ V &= V_R \quad \text{at infinity.} \end{aligned} \tag{2}$$

By putting in equation (1)

$$V = V_R+\exp\left(-\frac{Ux}{2D}\right)F(x,y) \tag{3}$$

we obtain

$$\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}-\frac{U^2}{4D^2}F=0 \tag{4}$$

or, in polar coordinates,

$$\frac{\partial^2 F}{\partial r^2}+\frac{1}{r}\frac{\partial F}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}-\frac{U^2}{4D^2}F=0. \tag{5}$$

By writing

$$F = R(r)\Theta(\theta)$$

we obtain

$$\frac{d^2\Theta}{d\theta^2}+n^2\Theta=0 \tag{6}$$

and

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-\left(\frac{U^2r^2}{4D^2}+n^2\right)R=0 \tag{7}$$

where n is a constant.

Since the solution must be an even single-valued function of θ, n is an integer, and the solution of (6) is

$$\Theta = A_n \cos n\theta. \tag{8}$$

The solution of (7) for integral n, which remains finite as r tends to infinity, is

$$R = B_n K_n(z), \tag{9}$$

where

$$K_{-n}(z) = K_n(z)$$

is the modified Bessel function of the second kind of order n, and where

$$\frac{Ur}{2D}=z \quad \text{and} \quad \frac{Ur_0}{2D}=z_0. \tag{10}$$

The solution of (1) is then

$$V = V_R + e^{-z\cos\theta} \sum_{n=-\infty}^{\infty} E_n K_n(z)\cos n\theta, \tag{11}$$

where the coefficients E_n are to be determined from the boundary condition at $r = r_0$.

On replacing z by $-iz$ in the expansion formulae for

$$e^{iz\cos\theta}$$

(Morse and Feshbach (**2**), p. 1322), we obtain

$$e^{z\cos\theta} = \sum_{n=-\infty}^{\infty} I_n(z)\cos n\theta, \tag{12}$$

where $$I_{-n}(z) = I_n(z)$$

is the modified Bessel function of the first kind of order n.

Applying the boundary condition at $r = r_0$ to (11), we obtain

$$(V_0 - V_R)\exp(z_0\cos\theta) = \sum_{n=-\infty}^{\infty} E_n K_n(z_0)\cos n\theta. \tag{13}$$

Comparing coefficients of $\cos n\theta$ in (12) and (13) we obtain

$$E_n = (V_0 - V_R)\frac{I_n(z_0)}{K_n(z_0)}. \tag{14}$$

On substituting (14) into (11) we obtain, finally,

$$V = V_R + (V_0 - V_R)e^{-z\cos\theta} \sum_{n=-\infty}^{\infty} \frac{I_n(z_0)}{K_n(z_0)} K_n(z)\cos n\theta, \tag{15}$$

which is the solution of (1) subject to the boundary conditions (2). For large values of z, equation (15) has the asymptotic form

$$\frac{V - V_R}{V_0 - V_R} \sim e^{-z\cos\theta}\left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} \frac{I_n(z_0)}{K_n(z_0)} e^{-z}\cos n\theta. \tag{16}$$

Now in the region $\frac{1}{2}\pi \leqslant \theta \leqslant \frac{3}{2}\pi$

$$z\cos\theta = -\frac{U|x|}{2D},$$

whence (16) becomes

$$\frac{V - V_R}{V_0 - V_R} \sim \exp\left[-\left(z - \frac{U|x|}{2D}\right)\right]\left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} \frac{I_n(z_0)}{K_n(z_0)}\cos n\theta. \tag{17}$$

In the 'wake' left behind,

$$|x| \to \infty \quad \text{and} \quad y \text{ is finite.}$$

Therefore $$z^{\frac{1}{2}} \sim \frac{U^{\frac{1}{2}}|x|^{\frac{1}{2}}}{(2D)^{\frac{1}{2}}} \tag{18}$$

and $$z - \frac{U|x|}{2D} \sim \frac{Uy^2}{4D|x|}. \tag{19}$$

Substituting (18) and (19) into equation (17), we obtain

$$\frac{V-V_R}{V_0-V_R} \sim \frac{B}{|x|^{\frac{1}{2}}}\exp\left(-\frac{Uy^2}{4D|x|}\right), \tag{20}$$

where B is a function of Z_0. It may be seen that (20) is an expression symmetrical about the x-axis, and at the centre of the 'wake' ($y = 0$), the temperature is

$$\frac{V-V_R}{V_0-V_R} = \frac{B}{|x|^{\frac{1}{2}}}. \tag{21}$$

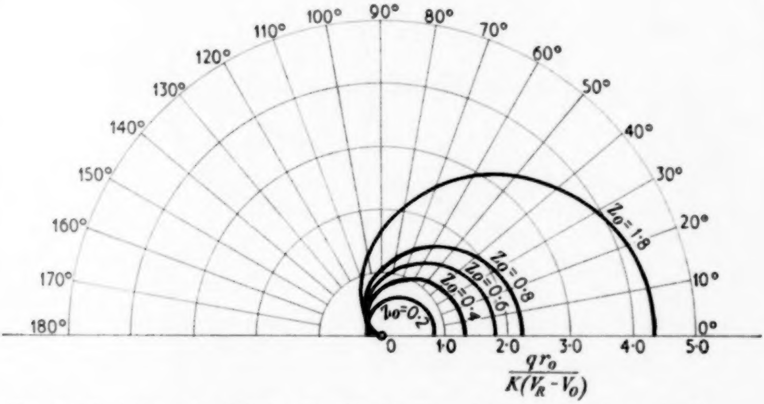

FIG. 2. Variation of $qr_0/K(V_R-V_0)$ with θ. (There is symmetry about the line $\theta = 0$, $\theta = \pi$.)

The amount of heat entering unit length of the cavity in unit time at the point (r_0, θ) is given by the formula

$$q = K\frac{\partial V}{\partial r} \quad \text{evaluated at } r = r_0.$$

Differentiating (15) with respect to r and setting $r = r_0$, we have

$$q = \frac{KU\cos\theta}{2D}(V_R-V_0)\exp(-z_0\cos\theta)\sum_{n=-\infty}^{\infty} I_n(z_0)\cos n\theta -$$
$$-\frac{KU(V_R-V_0)}{2D}\exp(-z_0\cos\theta)\sum_{n=-\infty}^{\infty}\frac{I_n(z_0)}{K_n(z_0)}K_n'(z_0)\cos n\theta \tag{22}$$

where primes denote differentiation with respect to z.

Equation (22) may be reduced by means of the Wronskian relation

$$I_m'(z_0)K_m(z_0)-I_m(z_0)K_m'(z_0) = \frac{1}{z_0} \tag{23}$$

and the relationship

$$2I'_m(z_0) = I_{m-1}(z_0)+I_{m+1}(z_0) \tag{24}$$

to

$$q = \frac{K(V_R-V_0)}{r_0}\exp(-z_0\cos\theta)\sum_{n=-\infty}^{\infty}\frac{\cos n\theta}{K_n(z_0)}. \tag{25}$$

A graph of the variation of $qr_0/K(V_R-V_0)$ with θ is shown in Fig. 2 for various values of z_0.

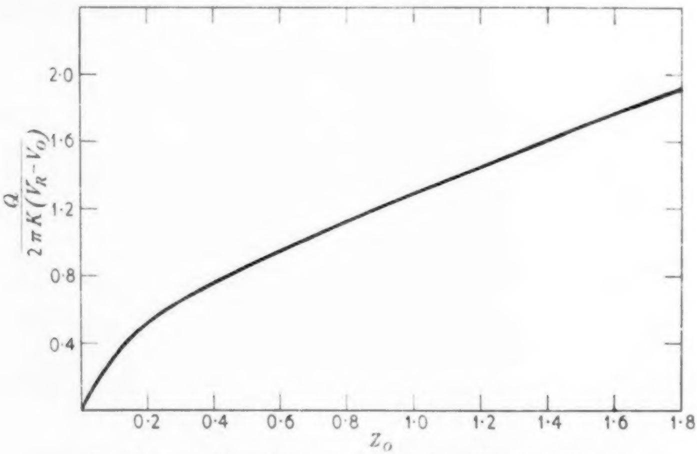

FIG. 3. Variation of the quantity $Q/2\pi K(V_R-V_0)$ with z_0.

The total quantity of heat entering the cavity per unit length per unit time is given by

$$Q = \int_0^{2\pi} qr_0\,d\theta \tag{26}$$

$$= 2\pi K(V_R-V_0)\sum_{n=-\infty}^{\infty}(-1)^n\frac{I_n(z_0)}{K_n(z_0)} \tag{27}$$

and the variation of $Q/2\pi K(V_R-V_0)$ with z_0 is shown in Fig. 3.

PART II

3. Introduction

We now consider the extension of the problem of Part I to the case in which the cross-section of the cavity is an ellipse with its major axis $2c$ in the direction of motion; the minor axis is $2d$ as indicated in Fig. 4.

4. Solution

Introducing elliptical coordinates defined by

$$x = a\cosh s\cos t, \qquad y = a\sinh s\sin t,$$

equation (4) becomes

$$\frac{\partial^2 F}{\partial t^2}+\frac{\partial^2 F}{\partial s^2}-2q[\cosh 2s-\cos 2t]F = 0, \tag{28}$$

where $$q = U^2a^2/16D^2.$$

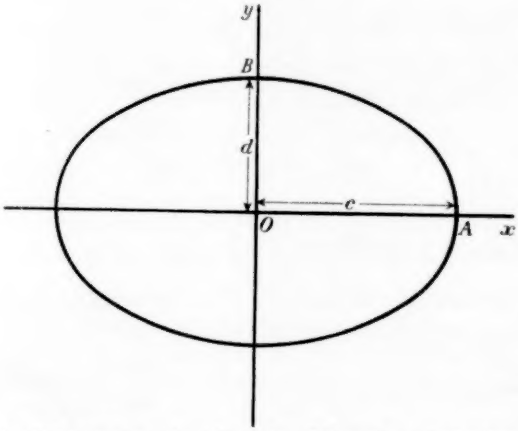

FIG. 4. $OA = c$ = semi-major axis. $OB = d$ = semi-minor axis. Motion of the cavity is in the direction Ox.

By putting $F = S(s)T(t)$ we obtain:

$$\frac{d^2T}{dt^2}+[2q\cos 2t+b]T = 0 \tag{29}$$

and $$\frac{d^2S}{ds^2}-[2q\cosh 2s+b]S = 0, \tag{30}$$

where b is a constant.

Since the solution must be an even periodic function of t, the solution of (29) is given by

$$T = A_n \operatorname{ce}_n(t, -q) \quad (n \text{ integral}), \tag{31}$$

where $\operatorname{ce}_n(t, -q)$ is an even Mathieu function of the first kind in the notation of McLachlan (**3**).

A solution of (30) which tends to zero as s tends to infinity is given by

$$S = B_n \operatorname{Fek}_n(s, -q). \tag{32}$$

The solution of (1) becomes

$$V = V_R+e^{-\omega\cosh s\cos t}\sum_{n=-\infty}^{\infty} E_n \operatorname{Fek}_n(s, -q)\operatorname{ce}_n(t, -q), \tag{33}$$

where $$Ua/2D = \omega$$

and the coefficients E_n are to be determined from the boundary condition

at $s = s_0$. Substituting in (33) we obtain

$$(V_0 - V_R)e^{\omega \cosh s_0 \cos t} = \sum_{n=-\infty}^{\infty} E_n \operatorname{Fek}_n(s_0, -q)\operatorname{ce}_n(t, -q). \tag{34}$$

Since
$$[\operatorname{ce}_n(t, -q)]$$
is an orthogonal set in the range $0 \leqslant t \leqslant 2\pi$, we have

$$E_n = \frac{(V_0 - V_R)[1+\delta(n)]}{2\operatorname{Fek}_n(s_0, -q)\mathscr{M}_n^e} \int_0^{2\pi} \exp(\omega \cosh s_0 \cos t)\operatorname{ce}_n(t, -q)\, dt, \tag{35}$$

where
$$\mathscr{M}_n^e = \int_0^{2\pi} \operatorname{ce}_n^2(t, -q)\, dt,$$

and
$$\delta(x) = \begin{cases} 1 & (x = 0), \\ 0 & (x \neq 0). \end{cases}$$

Following McLachlan (**3**) (section 2.18, equations (2) and (3)), and integrating term by term, we obtain the formulae

$$\int_0^{2\pi} \exp(\omega \cosh s_0 \cos t)\operatorname{ce}_{2n}(t, -q)\, dt = 2\pi(-1)^n \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} I_{2r}(\omega \cosh s_0) \tag{36}$$

and

$$\int_0^{2\pi} \exp(\omega \cosh s_0 \cos t)\operatorname{ce}_{2n+1}(t, -q)\, dt$$
$$= 2\pi(-1)^n \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{(2n+1)} I_{2r+1}(\omega \cosh s_0). \tag{37}$$

The values of E_n are given, therefore, by the formulae

$$E_{2n} = \frac{\pi(V_0 - V_R)[1+\delta(n)]}{\mathscr{M}_{2n}^e \operatorname{Fek}_{2n}(s_0, -q)}(-1)^n \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} I_{2r}(\omega \cosh s_0) \tag{38}$$

and

$$E_{2n+1} = \frac{\pi(V_0 - V_R)(-1)^n}{\mathscr{M}_{2n+1}^e \operatorname{Fek}_{2n+1}(s_0, -q)} \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{(2n+1)} I_{2r+1}(\omega \cosh s_0). \tag{39}$$

The temperature V at any point, (s, t), in the medium is given by (33) where we substitute (38) and (39) for E_{2n} and E_{2n+1}.

5. Flow of heat

The flow of heat into the cavity per unit length per unit time is given by

$$Q = K \int_0^{2\pi} \frac{\partial V}{\partial s}\, dt \tag{40}$$

evaluated at $s = s_0$. We obtain, on differentiating (33), and placing $s = s_0$

in the result, the formula

$$\frac{\partial V}{\partial s} = -\omega \sinh s_0 \cos t \exp(-\omega \cosh s_0 \cos t) \sum_{n=-\infty}^{\infty} E_n \operatorname{Fek}_n(s_0, -q) \operatorname{ce}_n(t, -q) +$$

$$+\exp(-\omega \cosh s_0 \cos t)\omega \sinh s_0 \sum_{n=-\infty}^{\infty} E_n \operatorname{Fek}'_n(s_0, -q) \operatorname{ce}_n(t, -q) \quad (41)$$

where primes denote differentiation with respect to $\omega \cosh s$.

Substituting values of $\operatorname{ce}_n(t, -q)$ given by formulae of McLachlan (**3**) (section 2.18) quoted above, and using the relation

$$2 \cos t \cos nt = \cos(n+1)t + \cos(n-1)t \quad (42)$$

and (23), we obtain

$$Q = 2\pi K \omega \sinh s_0 \times$$

$$\times \sum_{n=-\infty}^{\infty} E_{2n} \operatorname{Fek}_{2n}(s_0, -q)(-1)^n \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} I'_{2r}(\omega \cosh s_0) -$$

$$-2\pi K \omega \sinh s_0 \times$$

$$\times \sum_{n=-\infty}^{\infty} E_{2n+1} \operatorname{Fek}_{2n+1}(s_0, -q)(-1)^n \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{(2n+1)} I'_{2r+1}(\omega \cosh s_0) +$$

$$+2\pi K \omega \sinh s_0 \times$$

$$\times \sum_{n=-\infty}^{\infty} E_{2n} \operatorname{Fek}'_{2n}(s_0, -q)(-1)^n \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} I_{2r}(\omega \cosh s_0) -$$

$$-2\pi K \omega \sinh s_0 \times$$

$$\times \sum_{n=-\infty}^{\infty} E_{2n+1} \operatorname{Fek}'_{2n+1}(s_0, -q)(-1)^n \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{2n+1} I_{2r+1}(\omega \cosh s_0). \quad (43)$$

We will now verify that in the case when the ellipse tends to a circle, equations (33) and (43) reduce to (15) and (27) respectively.

The limit is approached in the following way. As $a \to 0$, s and s_0 both $\to \infty$ in such a way that

$$\left.\begin{aligned} a \cosh s \to a \sinh s \to r \\ a \cosh s_0 \to a \sinh s_0 \to r_0 \end{aligned}\right\} \quad (44)$$

and

$$\operatorname{ce}_n(t, -q) \to \cos n\theta, \quad (45)$$

where θ is the angle which the radius vector makes with the direction of motion, and

$$\mathscr{M}_n^e \to (1+\delta(n))\pi. \quad (46)$$

Also, according to McLachlan (**3**) (p. 369, equations (9) and (10))

$$\operatorname{Fek}_{2n}(s_0, -q) \to p'_{2n}\, \pi^{-1} K_{2n}(z_0), \quad (47)$$

$$\operatorname{Fek}_{2n+1}(s_0, -q) \to s'_{2n+1}\, \pi^{-1} K_{2n+1}(z_0). \quad (48)$$

From (47) and (48)

$$\operatorname{Fek}'_{2n}(s_0, -q) \to p'_{2n}\, \pi^{-1} K'_{2n}(z_0) \quad (49)$$

and

$$\operatorname{Fek}'_{2n+1}(s_0, -q) \to s'_{2n+1}\, \pi^{-1} K'_{2n+1}(z_0), \quad (50)$$

where in (49) and (50) primes denote differentiation with respect to the argument. From equations (44) to (48) we obtain

$$E_{2n} \to \frac{(V_0-V_R)I_{2n}(z_0)}{p'_{2n}\,\pi^{-1}K_{2n}(z_0)} \tag{51}$$

and

$$E_{2n+1} \to \frac{(V_0-V_R)I_{2n+1}(z_0)}{s'_{2n+1}\,\pi^{-1}K_{2n+1}(z_0)}. \tag{52}$$

Using equations (10), (44), (51), and (52), equation (33) becomes

$$V = V_R+(V_0-V_R)e^{-z\cos\theta}\sum_{n=-\infty}^{\infty}\frac{I_n(z_0)}{K_n(z_0)}K_n(z)\cos n\theta. \tag{15}$$

Using equations (44) and (47) to (52), equation (43) becomes

$$\begin{aligned} Q &= 2\pi K(V_0-V_R)z_0\Big[\sum_{n=-\infty}^{\infty} I_{2n}(z_0)I'_{2n}(z_0)-\sum_{n=-\infty}^{\infty} I_{2n+1}(z_0)I'_{2n+1}(z_0)+ \\ &\qquad +\sum_{n=-\infty}^{\infty}\frac{I^2_{2n}(z_0)K'_{2n}(z_0)}{K_{2n}(z_0)}-\sum_{n=-\infty}^{\infty}\frac{I^2_{2n+1}(z_0)K'_{2n+1}(z_0)}{K_{2n+1}(z_0)}\Big] \\ &= 2\pi K(V_0-V_R)z_0\Big[\sum_{n=-\infty}^{\infty}(-1)^n\frac{I_n(z_0)}{K_n(z_0)}\{I_n(z_0)K'_n(z_0)+I'_n(z_0)K_n(z_0)\}\Big]. \end{aligned} \tag{53}$$

But

$$\begin{aligned} &\sum_{n=-\infty}^{\infty}(-1)^nI_n(z_0)I'_n(z_0) \\ &\quad = \tfrac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^nI_n(z_0)[I_{n-1}(z_0)+I_{n+1}(z_0)] \\ &\quad = \tfrac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^nI_n(z_0)I_{n+1}(z_0)+\tfrac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^nI_n(z_0)I_{n-1}(z_0) \\ &\quad = \tfrac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^nI_n(z_0)I_{n+1}(z_0)-\tfrac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^nI_n(z_0)I_{n+1}(z_0) \\ &\quad = 0. \end{aligned} \tag{54}$$

Therefore

$$\begin{aligned} &\sum_{n=-\infty}^{\infty}(-1)^n\frac{I_n(z_0)}{K_n(z_0)}\{I_n(z_0)K'_n(z_0)+I'_n(z_0)K_n(z_0)\} \\ &\quad = \sum_{n=-\infty}^{\infty}(-1)^n\frac{I^2_n(z_0)K'_n(z_0)}{K_n(z_0)}+\sum_{n=-\infty}^{\infty}(-1)^nI_n(z_0)I'_n(z_0) \\ &\quad = \sum_{n=-\infty}^{\infty}(-1)^n\frac{I^2_n(z_0)K'_n(z_0)}{K_n(z_0)}-\sum_{n=-\infty}^{\infty}(-1)^nI_n(z_0)I'_n(z_0) \\ &\quad = \sum_{n=-\infty}^{\infty}(-1)^n\frac{I_n(z_0)}{K_n(z_0)}\{I_n(z_0)K'_n(z_0)-I'_n(z_0)K_n(z_0)\}. \end{aligned} \tag{55}$$

On using equation (23), we obtain

$$\sum_{n=-\infty}^{\infty} (-1)^n \frac{I_n(z_0)}{K_n(z_0)} \{I_n(z_0)K_n'(z_0) + I_n'(z_0)K_n(z_0)\} = -\sum_{n=-\infty}^{\infty} (-1)^n \frac{I_n(z_0)}{z_0\, K_n(z_0)}. \tag{56}$$

Therefore, substituting (56) into (53) we obtain

$$Q = 2\pi K(V_R - V_0) \sum_{n=-\infty}^{\infty} (-1)^n \frac{I_n(z_0)}{K_n(z_0)}. \tag{27}$$

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ON THE SOLUTION OF SOME AXISYMMETRIC BOUNDARY VALUE PROBLEMS BY MEANS OF INTEGRAL EQUATIONS

I. SOME ELECTROSTATIC AND HYDRODYNAMIC PROBLEMS FOR A SPHERICAL CAP

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SUMMARY

Various boundary value problems for a spherical cap in electrostatics and in the theories of the potential flow of a perfect fluid and the Stokes flow of a viscous fluid are considered.

1. Introduction

In this paper we derive the solutions of certain boundary value problems for a spherical cap. In section 2 the electrostatic potential due to a perfectly conducting spherical cap, maintained at a given potential which is symmetrical about the axis of the cap, is found as the real part of a certain integral, the expression of which is similar to that found by Green and Zerna (**1**) in the corresponding problem for a circular disk. It may be noted that this potential can also be found by first determining the surface charge density on the cap in a manner similar to that by which Copson (**2**) found the surface charge density on a circular disk. The method given in this paper however gives the potential directly. The potentials due to two given distributions on the cap are then deduced. In section 3 we show that the stream-function for the irrotational axisymmetric motion of a perfect fluid about a spherical cap, the stream-function being specified on the cap, can be expressed as the imaginary part of a certain integral and this is applied to find the stream-function for a uniform motion of fluid past the cap. Using an analogy derived by the author (**3**), we consider in section 4 the slow steady rotation of a cap about its axis in viscous fluid and find the couple required to maintain the steady motion of the cap.

2. The electrostatic potential problem for a spherical cap

We require the electrostatic potential due to a thin spherical cap, which is maintained at a given potential symmetrical about its axis. We use spherical polar coordinates (r, θ, ϕ) referred to the centre of the sphere $r = a$ as origin and the axis of the cap as polar axis, the cap then being

given by $r = a$, $0 \leqslant \theta \leqslant \alpha$. The electrostatic potential $V(r,\theta)$ is a harmonic function, continuously differentiable everywhere except possibly at the edge of the cap, and is $O(r^{-1})$ at large distances from the cap. Its value on the cap is given as a known even function $f(\theta)$ so that

$$V = f(\theta) \quad \text{when} \quad r = a \quad (0 \leqslant \theta \leqslant \alpha). \tag{2.1}$$

We now show that such a function $V(r,\theta)$ is given by

$$V(r,\theta) = \tfrac{1}{2}a \int_{-\alpha}^{\alpha} \frac{g(\eta)\sec\frac{1}{2}\eta \, d\eta}{(r^2e^{i\eta}+a^2e^{-i\eta}-2ar\cos\theta)^{\frac{1}{2}}}, \tag{2.2}$$

where $g(\eta)$ is a real continuous even function of η and where for $r \geqslant a$

$$\sqrt{(r^2e^{i\eta}+a^2e^{-i\eta}-2ar\cos\theta)} = \rho e^{\frac{1}{2}i\tau}, \tag{2.3}$$

$$\rho \geqslant 0, \qquad 0 \leqslant \tau \leqslant \pi \quad \text{for} \quad 0 \leqslant \eta \leqslant \alpha,$$
$$-\pi < \tau \leqslant 0 \quad \text{for} \quad -\alpha \leqslant \eta < 0,$$

and for $r < a$

$$\sqrt{(r^2e^{i\eta}+a^2e^{-i\eta}-2ar\cos\theta)} = \rho e^{-\frac{1}{2}i\tau}, \tag{2.4}$$

$$\rho \geqslant 0, \qquad 0 \leqslant \tau \leqslant \pi \quad \text{for} \quad 0 \leqslant \eta \leqslant \alpha,$$
$$-\pi < \tau \leqslant 0 \quad \text{for} \quad -\alpha \leqslant \eta < 0.$$

When $r = a$, we have

$$\sqrt{(r^2e^{i\eta}+a^2e^{-i\eta}-2ar\cos\theta)} = a\sqrt{(2\cos\eta-2\cos\theta)} \quad (\theta \geqslant |\eta|),$$

while

$$\begin{aligned}\sqrt{(r^2e^{i\eta}+a^2e^{-i\eta}-2ar\cos\theta)} &\\ &= \pm ia\sqrt{(2\cos\theta-2\cos\eta)} \quad (\theta < |\eta|),\\ &\qquad\qquad \text{if } r \to a \text{ through values greater than } a,\\ &= \mp ia\sqrt{(2\cos\theta-2\cos\eta)} \quad (\theta < |\eta|),\\ &\qquad\qquad \text{if } r \to a \text{ through values less than } a,\end{aligned} \tag{2.5}$$

the upper sign holding for $0 \leqslant \eta \leqslant \alpha$ and the lower for $-\alpha \leqslant \eta \leqslant 0$. There is thus a discontinuity in the square root at $r = a$ $(\theta < |\eta|)$. We further note that the integral in (2.2) is to be interpreted as a Cauchy integral at the point $r = a$, $\theta = 0$.

Since $g(\eta)$ is an even function, $V(r,\theta)$ as defined by (2.2) is real and is $O(r^{-1})$ for large r. Further, the integrand of $V(r,\theta)$ is a continuously differentiable harmonic function of r and θ at all points where its denominator is not zero, that is, wherever

$$r^2e^{i\eta}+a^2e^{-i\eta}-2ar\cos\theta \neq 0,$$

this being so everywhere except on the cap $r = a$ $(0 \leqslant \theta \leqslant \alpha)$, hence

$V(r,\theta)$ is harmonic and continuously differentiable everywhere except possibly on the cap $r = a$ $(0 \leqslant \theta \leqslant \alpha)$.

We next show that $V(r,\theta)$ is continuous for normal approach to the cap as $r \to a$ through values greater than a, a similar proof to the following holding when $r \to a$ through values less than a. We suppose that $0 < \theta \leqslant \alpha$ and note the relation

$$\begin{aligned}|r^2e^{i\eta}+a^2e^{-i\eta}-2ar\cos\theta|^2 \\ &= 4a^2r^2(\cos\eta-\cos\theta)^2+(r-a)^2[(r-a)^2+4ar(1-\cos\theta\cos\eta)] \\ &\geqslant 4a^2r^2(\cos\eta-\cos\theta)^2+(r-a)^4.\end{aligned}$$

Thus for $r \geqslant a$ we have

$$|r^2e^{i\eta}+a^2e^{-i\eta}-2ar\cos\theta| \geqslant 2a^2|\cos\eta-\cos\theta|, \tag{2.6}$$

and hence obtain

$$\left|\frac{g(\eta)\sec\frac{1}{2}\eta}{(r^2e^{i\eta}+a^2e^{-i\eta}-2ar\cos\theta)^{\frac{1}{2}}}\right| \leqslant \frac{|g(\eta)|\sec\frac{1}{2}\eta}{a\sqrt{|2\cos\eta-2\cos\theta|}}.$$

By the theorem of dominated convergence we have that, as $r \to a$ through values less than a,

$$\int\limits_{-\alpha}^{\alpha}\frac{g(\eta)\sec\frac{1}{2}\eta\,d\eta}{\sqrt{(r^2e^{i\eta}+a^2e^{-i\eta}-2ar\cos\theta)}} \to \int\limits_{-\alpha}^{\alpha}\frac{g(\eta)\sec\frac{1}{2}\eta\,d\eta}{a\sqrt{(2\cos\eta-2\cos\theta)}}, \tag{2.7}$$

provided $g(\eta)$ is bounded, this holding at all points on the cap except $\theta = 0$. This failure at the point $r = a, \theta = 0$, does not however invalidate the solution, since it is shown later that $V(r,\theta)$ tends to a finite limit at this point. The square root on the right of (2.7) is to be interpreted as the limit of that on the left in accordance with (2.3) and (2.5).

We further require that $V(r,\theta)$ be continuously differentiable on either side of the cap.† The zeros of the function

$$r^2e^{i\zeta}+a^2e^{-i\zeta}-2ar\cos\theta \quad (\zeta = \eta+i\xi),$$

regarded as a function of ζ, lie in the upper half ζ-plane when $r > a$ and in the lower half plane when $r < a$. We can therefore displace the path of integration in (2.2) from $-\alpha$ to α away from the real axis into the lower and upper half ζ-planes for $r > a$ and $r < a$ respectively, provided we assume $g(\zeta)$ to be an analytic function in a simply-connected domain which contains both the old and the new paths of integration. In the new regions of definition of $V(r,\theta)$ the integrands are continuously differentiable functions of r and θ, other than on the edge of the cap $r = a, \theta = \alpha$. Therefore the derivatives of $V(r,\theta)$ with respect to r and θ exist and are continuous in the neighbourhood of the cap, and can be calculated by differentiation

† I am grateful to a referee for certain suggestions in regard to this and the preceding parts of the proof.

under the integral sign. Thus $V(r,\theta)$ is harmonic in this neighbourhood and tends to finite limits as the point (r,θ) approaches a point on the cap, including the point $(a,0)$, in any manner through values of r greater or less than a, these limits being equal by the argument of the preceding paragraph. Since $V(r,\theta)$ admits this harmonic extension through the cap, it must in its original region of definition be continuously differentiable on either side of the cap, except at its edge.

Using (2.3) and (2.5), we have from (2.7) on $r = a$ $(0 \leqslant \theta \leqslant \alpha)$,

$$V(a,\theta) = \int\limits_0^\theta \frac{g(\eta)\sec\frac{1}{2}\eta\, d\eta}{\sqrt{(2\cos\eta - 2\cos\theta)}} = f(\theta). \tag{2.8}$$

Making the substitutions $t = \tan\frac{1}{2}\eta$, $s = \tan\frac{1}{2}\theta$, and writing

$$g(\eta) \equiv G(\tan\tfrac{1}{2}\eta), \qquad f(\theta) \equiv F(\tan\tfrac{1}{2}\theta),$$

we reduce this equation to

$$\int\limits_0^s \frac{G(t)\, dt}{\sqrt{(s^2-t^2)}} = \frac{F(s)}{\sqrt{(1+s^2)}} \quad (0 \leqslant s \leqslant \tan\tfrac{1}{2}\alpha). \tag{2.9}$$

This equation is now of the form considered by Green and Zerna (**1**) and, if $f(\theta)$ is continuous for $0 \leqslant \theta \leqslant \alpha$, its solution is

$$G(t) = \frac{2}{\pi}\frac{d}{dt}\int\limits_0^t \frac{sF(s)\, ds}{[(t^2-s^2)(1+s^2)]^{\frac{1}{2}}} \quad (0 \leqslant t \leqslant \tan\tfrac{1}{2}\alpha). \tag{2.10}$$

When $G(t)$ has been determined from (2.10), the value of $V(r,\theta)$ follows from (2.2).

When $\alpha = \pi$, the cap becomes a sphere of radius a. The expression (2.2) still holds for either the region $r \geqslant a$ or the region $r \leqslant a$. It can be shown from (2.10) that, if $f(\theta)$ is continuously differentiable for $0 \leqslant \theta \leqslant \pi$, $g(\eta)$ is $O(\cos^2\frac{1}{2}\eta)$ for $\pi - \epsilon < \eta \leqslant \pi$, where ϵ is a small positive quantity. It then follows that $V(r,\theta)$ is continuously differentiable at all points for which $r \geqslant a$ (or $r \leqslant a$) including the point $r = a$, $\theta = \pi$.

The total charge Σ on the cap is the limit, as $r \to \infty$, of (**4**)

$$\tfrac{1}{2}ra\int\limits_{-\alpha}^{\alpha} \frac{g(\eta)\sec\frac{1}{2}\eta\, d\eta}{(r^2e^{i\eta}+a^2e^{-i\eta}-2ar\cos\theta)^{\frac{1}{2}}},$$

so that

$$\Sigma = a\int\limits_0^\alpha g(\eta)\, d\eta. \tag{2.11}$$

As illustrations of these results we consider the following problems.

(*a*) *Spherical cap at constant potential.* For a spherical cap at a constant potential U we have $f(\theta) = U$ and from (2.10) we obtain

$$G(t) = \frac{2U}{\pi(1+t^2)},$$

or

$$g(\eta) = \frac{2U}{\pi}\cos^2\tfrac{1}{2}\eta.$$

On substituting in (2.2) and integrating, we obtain

$$V(r,\theta) = \frac{U}{\pi r}(r\gamma + a\gamma'), \tag{2.12}$$

where

$$\frac{\sin\gamma}{a} = \frac{\sin\gamma'}{r} = \frac{2\sin\alpha}{r_1+r_2}$$

and $\quad r_1^2 = r^2+a^2-2ar\cos(\alpha-\theta), \qquad r_2^2 = r^2+a^2-2ar\cos(\alpha+\theta).$

The angle $\gamma \equiv \gamma(r,\theta)$ is such that $0 < \gamma \leqslant \frac{1}{2}\pi$ at all points (r,θ,ϕ) other than those which lie in that segment of the sphere $r = a$ bounded by the surface of the cap and the plane, $r\cos\theta = a\cos\alpha$, containing its edge, for which points $\frac{1}{2}\pi < \gamma < \pi$. Further, if S is the sphere

$$r^2\cos\alpha - ar\cos\theta = 0,$$

through the edge of the cap and the centre of the sphere $r = a$, the angle $\gamma' \equiv \gamma'(r,\theta)$ is such that $\frac{1}{2}\pi < \gamma' < \pi$ for those points (r,θ,ϕ) exterior to the sphere $r = a$ but interior to S for $\alpha < \frac{1}{2}\pi$ and exterior to S for $\alpha > \frac{1}{2}\pi$. For all other points $0 < \gamma' \leqslant \frac{1}{2}\pi$.

From (2.11) the total charge Σ on the cap is given as

$$\Sigma = \frac{Ua}{\pi}(\alpha + \sin\alpha);$$

so the capacity C of the cap is given by

$$C = \frac{\Sigma}{U} = \frac{a(\alpha+\sin\alpha)}{\pi}.$$

(*b*) *Spherical cap in a uniform field.* If an earthed spherical cap is placed in a uniform field E parallel to the axis of the cap, we write

$$V = -Er\cos\theta + V_1,$$

so that on the cap the perturbation part V_1 of V satisfies the condition

$$V_1 = Ea\cos\theta \quad (0 \leqslant \theta \leqslant \alpha).$$

Hence

$$F(s) = Ea\left(\frac{1-s^2}{1+s^2}\right),$$

and from (2.10) we find that

$$G(t) = \frac{2Ea(1-3t^2)}{\pi(1+t^2)^2},$$

so that
$$g(\eta) = \frac{2Ea}{\pi}\cos\tfrac{1}{2}\eta\cos\tfrac{3}{2}\eta.$$

On substituting in (2.2) and integrating, we obtain
$$V(r,\theta) = -Ez+\frac{E}{\pi}\left[z\gamma+(a\cos\alpha-z)\tan\gamma+\frac{a^3z\gamma'}{r^3}+\frac{a^2(r^2\cos\alpha-az)\tan\gamma'}{r^3}\right],$$
where $z = r\cos\theta$ and the angles γ and γ' are interpreted as in (2.12).

These results are in agreement with those obtained by alternative methods by Ferrers (**5**), Gallop (**6**), Basset (**7**), and Szegö (**8**).

3. The stream-function for the motion of a perfect fluid past a spherical cap

In an irrotational axisymmetrical motion of a perfect fluid the Stokes stream-function Ψ satisfies the equation
$$\Lambda^2\Psi = \left[\frac{\partial^2}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]\Psi = 0, \tag{3.1}$$
together with the condition that Ψ is a constant over any rigid boundary present in the flow.

We now suppose that the stream-function $\Psi(r,\theta)$ for such a flow past a spherical cap is given by
$$\Psi(r,\theta) = \psi_0(r,\theta)+\psi(r,\theta),$$
where $\psi_0(r,\theta)$ is the stream-function for the flow in an infinite unbounded fluid in the absence of the cap and $\psi(r,\theta)$ the perturbation stream-function, which must be added to $\psi_0(r,\theta)$ to make the boundary of the cap, $r = a$ $(0 \leqslant \theta \leqslant \alpha)$, a stream-sheet. We require to determine $\psi(r,\theta)$ at a general point of the fluid. On the boundary of the cap $\Psi(r,\theta)$ is constant and thus $\psi(r,\theta)$ is a given even function of θ, say $f(\theta)$, on this boundary. We have
$$\psi(r,\theta) = f(\theta) \quad \text{on} \quad r = a \quad (0 \leqslant \theta \leqslant \alpha), \tag{3.2}$$
where $f(\theta) = \text{constant } -\psi_0(a,\theta)$. Since the constant is at our disposal, we choose it so that $f(\theta) = 0$ at the point $r = a$, $\theta = 0$.

Since the velocity components q_r, q_θ, in the r- and θ-directions, given by
$$q_r = -\frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta}, \qquad q_\theta = \frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r},$$
are continuous at all points except possibly those on the edge of the cap, $\psi(r,\theta)$ must be continuously differentiable at these points. Further, since the introduction of the spherical cap into the flow can cause no additional total flow over the sphere at infinity, the velocity components derived from the perturbation stream-function are $O(r^{-3})$ at a large distance r from the origin and hence $\psi(r,\theta)$ is $O(r^{-1})$ at this distance.

We now show that a stream-function $\psi(r,\theta)$, satisfying these conditions, is given by

$$2i\psi(r,\theta) = \int_{-\alpha}^{\alpha} \frac{g(\eta)(re^{\frac{1}{2}i\eta}\cos\theta - ae^{-\frac{1}{2}i\eta})\,d\eta}{(r^2e^{i\eta}+a^2e^{-i\eta}-2ar\cos\theta)^{\frac{1}{2}}} - \frac{r}{a}\int_{-\alpha}^{\alpha} \frac{g(\eta)(ae^{-\frac{1}{2}i\eta}\cos\theta - re^{\frac{1}{2}i\eta})\,d\eta}{(r^2e^{i\eta}+a^2e^{-i\eta}-2ar\cos\theta)^{\frac{1}{2}}}, \tag{3.3}$$

where $g(\eta)$ is a real continuous odd function of η and the square root $(r^2e^{i\eta}+a^2e^{-i\eta}-2ar\cos\theta)^{\frac{1}{2}}$ is interpreted as in (2.3), (2.4), and (2.5).

Since $g(\eta)$ is an odd function of η, the stream-function $\psi(r,\theta)$ given by (3.3) is a real function. The expression $\Lambda^2\psi$ can be evaluated at all points where the necessary differentiations can be carried out under the integral signs, that is, at all points where the integrands so obtained are continuous functions of (r,θ,η). The only points where this condition is not satisfied are points such that

$$r^2e^{\pm i\eta}+a^2e^{\mp i\eta}-2ar\cos\theta = 0,$$

that is, points for which $r = a$, $\eta = \pm\theta$. All such points lie on the cap $r = a$ $(0 \leqslant \theta \leqslant \alpha)$. Now the integrand of the first integral is readily seen to be a solution of (3.1) and the application of Butler's inversion theorem for a sphere (**9**) to this integrand shows that the integrand of the second integral is also a solution of (3.1). Hence $\psi(r,\theta)$ is a continuously differentiable function of r and θ and satisfies (3.1) everywhere except possibly on the cap $r = a$ $(0 \leqslant \theta \leqslant \alpha)$.

As for the electrostatic potential in section 2, we can show that $\psi(r,\theta)$ is continuously differentiable on either side of the cap except at its edge $r = a, \theta = \alpha$, and that the limits of the integrals in (3.3) as (r,θ) approaches a point on the cap are equal to the values of the integrals at that point.

We next note that, if the integrands in (3.3) be expanded as power series in a/r, we obtain

$$\psi(r,\theta) = \frac{a\sin^2\theta}{r}\int_0^{\alpha} g(\eta)\sin\eta(1+\cos\eta)\,d\eta + \text{terms in } \left(\frac{a}{r}\right)^2, \tag{3.4}$$

showing that $\psi(r,\theta)$ has the required order for large r. Using (2.5), we have on the cap

$$\psi(a,\theta) = 2(1+\cos\theta)\int_0^{\theta} \frac{g(\eta)\sin\frac{1}{2}\eta\,d\eta}{(2\cos\eta-2\cos\theta)^{\frac{1}{2}}} = f(\theta) \quad (r = a,\ 0 \leqslant \theta \leqslant \alpha).$$

Making the substitutions $t = \tan\frac{1}{2}\eta$, $s = \tan\frac{1}{2}\theta$, and writing

$$g(\eta) \equiv G(\tan\tfrac{1}{2}\eta), \qquad f(\theta) \equiv F(\tan\tfrac{1}{2}\theta),$$

as before, we obtain

$$\int_0^s \frac{tG(t)\,dt}{(1+t^2)\sqrt{(s^2-t^2)}} = \tfrac{1}{4}F(s)\sqrt{(1+s^2)} \quad (0 \leqslant s \leqslant \tan\tfrac{1}{2}\alpha), \tag{3.5}$$

which is again an integral equation of the type considered by Green and Zerna (**1**). If $f(\theta)$ is continuous, its solution is

$$\frac{tG(t)}{1+t^2} = \frac{1}{2\pi}\frac{d}{dt}\int_0^t \frac{s\sqrt{(1+s^2)}F(s)\,ds}{\sqrt{(t^2-s^2)}} \quad (0 \leqslant t \leqslant \tan\tfrac{1}{2}\alpha). \tag{3.6}$$

When $G(t)$ has been determined from this equation, $\psi(r,\theta)$ follows from (3.3). We note that, if $F(s)$ is given as a constant, $G(t)$ is $O(t^{-1})$ for small t. Hence it is necessary to choose the disposable constant in $F(s) \equiv f(\theta)$ to ensure that $F(s) = 0$ at the point $r = a$, $\theta = 0$. We then have $F(s)$ is $O(s^2)$ for small s and it follows from (3.6) that $G(t)$ is bounded at the origin.

The cap becomes a sphere when $\alpha = \pi$ and the expression (3.3) then applies to motions external or internal to the sphere. In this case it follows from (3.6) that, if $f(\theta)$ is $O(\cos^2\frac{1}{2}\theta)$ for $\pi-\epsilon < \theta \leqslant \pi$, where ϵ is a small positive quantity, $g(\eta)$ is $O(\cos\frac{1}{2}\eta)$ for $\pi-\delta < \eta \leqslant \pi$, where δ is a small positive quantity depending on ϵ. It can then be shown that $\psi(r,\theta)$ is continuously differentiable at all points for which $r \geqslant a$ (or $r \leqslant a$) including the point $r = a$, $\theta = \pi$. Now the stream-function $\psi_0(r,\theta)$ for the motion in the infinite unbounded fluid is constant between any two sources on the axis of the motion, the value of the constant changing by an amount $2m$ at such a source of strength m. For motions external to the sphere any such singularities lie in the region $r > a$ and thus $f(\theta)$ is zero at $r = a$, $\theta = \pi$, since it is zero at $r = a$, $\theta = 0$. It then follows that $f(\theta)$ is $O(\cos^2\frac{1}{2}\theta)$ near $\theta = \pi$. For motions internal to the sphere the total change in $\psi_0(r,\theta)$ along that part of the axis inside the sphere is $2\times$(total strength of the sources on the axis inside the sphere) and this must be zero since fluid is conserved inside the sphere. It then follows that $f(\theta)$ is zero at $r = a$, $\theta = \pi$, and is thus $O(\cos^2\frac{1}{2}\theta)$ near $\theta = \pi$.

We next give a corresponding result to the above for a circular disk of radius c with its centre on the axis of symmetry. If (ϖ, ϕ, z) are cylindrical polar coordinates with the centre of the disk as origin and the axis of symmetry as z-axis, we require a stream-function $\psi(\varpi, z)$ such that on the disk

$$\psi = f(\varpi) \quad (z = 0,\ 0 \leqslant \varpi \leqslant c),$$

where $f(\varpi)$ is a continuous even function of ϖ with $f(0) = 0$. Further, $\psi(\varpi, z)$ is a solution of (3.1), is continuously differentiable at all points

except those on the rim of the disk $z = 0$, $\varpi = c$, and is $O(r^{-1})$ at large distances r from the origin. It can be shown by a similar proof to that for the cap that $\psi(\varpi, z)$ is given by

$$2i\psi(\varpi, z) = \int\limits_{-c}^{c} \frac{g(t)(z+it)\,dt}{\{\varpi^2+(z+it)^2\}^{\frac{1}{2}}},$$

where $g(t)$ is a real continuous odd function of t, such that

$$tg(t) = \frac{2}{\pi}\frac{d}{dt}\int\limits_{0}^{t} \frac{\varpi f(\varpi)\,d\varpi}{\sqrt{(t^2-\varpi^2)}}.$$

The square root $\{\varpi^2+(z+it)^2\}^{\frac{1}{2}}$ is interpreted similarly to the square root occurring in (2.2).

We now give an illustration of the use of (3.3) and (3.6):

A spherical cap in a uniform stream. If a spherical cap is placed in a uniform stream of velocity U in the negative z-direction, the axis of the cap being taken as the z-axis, the perturbation stream-function $\psi(r, \theta)$ satisfies

$$\psi = f(\theta) = -\tfrac{1}{2}Ua^2\sin^2\theta \quad (r = a,\ 0 \leqslant \theta \leqslant \alpha).$$

Thus

$$F(s) = -\frac{2Ua^2s^2}{(1+s^2)^2},$$

and from (3.6) we find that

$$G(t) = -\frac{2Ua^2t}{\pi(1+t^2)},$$

or

$$g(\eta) = -\frac{Ua^2}{\pi}\sin\eta. \tag{3.7}$$

On using (3.3) we obtain

$$\psi(r, \theta) = \frac{U}{2\pi}\Big[-\gamma r^2\sin^2\theta-\gamma'\frac{a^3}{r}\sin^2\theta+(r\cos\theta+a)(a\cos\alpha-r\cos\theta)\tan\gamma+$$
$$+\frac{a}{r}(r+a\cos\theta)(r\cos\alpha-a\cos\theta)\tan\gamma'\Big], \tag{3.8}$$

where r_1, r_2, γ, and γ' have the same meanings as in (2.12).

4. The slow steady rotation of a spherical cap in a viscous fluid

If a solid of revolution rotates slowly about its axis with angular velocity Ω in a viscous fluid, the only non-zero velocity component is the component v in the ϕ-direction. The author showed in a recent paper (**3**) that solutions for v are given by

$$v = \frac{\psi}{r\sin\theta}, \tag{4.1}$$

where ψ is the stream-function for the same solid moving with constant velocity U along its axis in perfect fluid, provided U is replaced by -2Ω in ψ. This result can now be applied to give the solution for a rotating spherical cap. In fact (3.8), besides giving the perturbation stream-function for a uniform motion past the cap, also gives the stream-function for a translation of the cap with uniform velocity U in the positive z-direction. Replacing U by -2Ω and using (4.1), we can thus derive v from (3.8).

In order to find the couple Q required to maintain the rotation of the cap we use (3.4) to find that, at a large distance r from the origin, v has the form

$$v = \frac{a\sin\theta}{r^2}\int\limits_0^\alpha g(\eta)\sin\eta(1+\cos\eta)\,d\eta,$$

where U has been replaced by -2Ω in the function $g(\eta)$. If μ is the viscosity of the fluid, the couple Q required to maintain the motion is given by (**10**)

$$Q = -2\pi\mu\lim_{r\to\infty}\int\limits_0^\pi \sin^2\theta\, r^4\frac{\partial}{\partial r}\left(\frac{v}{r}\right)d\theta = 8\pi\mu a\int\limits_0^\alpha g(\eta)\sin\eta(1+\cos\eta)\,d\eta.$$

Hence from (3.7) we have

$$Q = 16\Omega\mu a^3\int\limits_0^\alpha \sin^2\eta(1+\cos\eta)\,d\eta = 4\Omega\mu a^3(2\alpha-\sin 2\alpha+\tfrac{4}{3}\sin^3\alpha).$$

If α tends to zero and a tends to infinity in such a way that $a\alpha$ tends to a finite limit c, the couple Q becomes that required to maintain the rotation of a circular disk of radius c, namely,

$$Q = \frac{32\mu\Omega c^3}{3}.$$

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ON THE EQUATION OF A SYNCHRONOUS MOTOR

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SUMMARY

The strongly non-linear differential equation $\ddot{x}+\alpha\dot{x}+\sin x = \alpha\eta$, which arises in the study of synchronous motors, is discussed for small values of α by the method of small parameters and by a generalization of the method of slowly varying amplitude and phase.

1. Introduction

THE oscillations of a self-starting synchronous motor are governed by the non-linear differential equation (**1**)

$$\ddot{x}+\alpha\dot{x}+\sin x = \beta, \tag{1}$$

where α and β are positive constants. The same equation also represents the motion of a pendulum which is acted on by a constant torque and opposed by viscous friction. The first mathematical study of the equation (1) is due to Tricomi (**2**), some of whose results are reported in Andronov and Chaikin (**3**) and other texts on non-linear mechanics. Among more recent investigations those of Amerio (**4**) and Bohm (**5**) may be especially noted. Amerio (**6**) has also generalized many properties of the equation (1) to a wide class of equations of the form

$$\ddot{x} = F(x, \dot{x}),$$

where F is periodic in x.

The question of greatest interest is the asymptotic behaviour of the solutions when $t \to +\infty$ and, in particular, the manner in which this behaviour depends on the initial conditions and on the values of the constants α, β. This question has been completely answered in a qualitative sense by the work of Tricomi and Amerio. It has been shown that there are two possible types of solution—the 'convergent' solutions which tend to a finite limit as $t \to +\infty$, and the 'divergent' solutions which behave asymptotically as the sum of a linear function and a periodic function. Divergent solutions exist if and only if the first-order equation

$$y\,dy/dx+\alpha y+\sin x = \beta, \tag{2}$$

which is obtained from (1) by putting $y = \dot{x}$, has a positive solution $y(x)$ of period 2π. If $\beta > 1$ all solutions are divergent. If $\beta < 1$ and if the ratio $\eta = \beta/\alpha$ is less than a critical value η_c, depending on β, then all solutions are convergent; if $\beta < 1$ and $\eta > \eta_c$ then both convergent and divergent solutions are present. The initial conditions which characterize

the different types of solution can be determined at once from a knowledge of the two solutions of (2) which pass through the singular point $(\pi - \sin^{-1}\beta, 0)$.

On the quantitative side all that is known concerns the critical value $\eta_c(\beta)$. It has been calculated to three decimal places for $\beta = 0(\cdot 1)1$ by Urabe (**7**) and Giger (**8**), by numerical integration of the first-order equation (2). The difficulty in obtaining further information has been attributed to the strongly non-linear character of the equation (1). In fact, since x is unrestricted, it is not permissible to replace $\sin x$ by x or $x - \frac{1}{6}x^3$, as one often does on other occasions.

In the technical applications, however, α is usually small and this suggests that we can treat (1) not as a quasi-linear equation, but as a quasi-conservative one. This is what is done here. It is shown that the rigorous method of small parameters of Poincaré (**9**) for the determination of periodic solutions can be applied to (2), and provides new information about the divergent solutions.† It is shown also that the heuristic method of slowly varying amplitude and phase, which has been developed by Kryloff and Bogoliuboff (**11**) for certain quasi-linear equations, can be extended to the equation (1), and leads to equations soluble in closed form. The results obtained in this way are in agreement with the known qualitative results.

2. The method of small parameters

2.1. Consider first of all the general equation

$$y\,dy/dx + g(x) = \alpha f(x,y), \tag{3}$$

where $g(x)$ is a continuous function of period ω such that

$$\int_0^\omega g(x)\,dx = 0,$$

and where $f(x,y)$ is a continuous function with period ω in x having a continuous partial derivative with respect to y. We wish to know if, for small values of α, this equation has a solution $y(x)$ which is positive and of period ω.

For $\alpha = 0$, (3) reduces to the elementary equation

$$Y\,dY/dx + g(x) = 0.$$

This has the positive periodic solutions

$$Y = Y(x,C) = 2^{\frac{1}{2}}[C - G(x)]^{\frac{1}{2}},$$

where

$$G(x) = \int_0^x g(\xi)\,d\xi \tag{4}$$

† The method of small parameters has already been used by Urabe (**10**), but in quite a different manner and not for the determination of the periodic solutions.

and C is any constant greater than $\max G(x)$ $(0 \leqslant x \leqslant \omega)$. Since Y takes the value $(2C)^{\frac{1}{2}}$ at $x = 0$, we have $C = \frac{1}{2}Y^2(0)$.

Suppose now that $\alpha \neq 0$, and let $y(x)$ be a solution of (3) which is defined and positive for $0 \leqslant x \leqslant \omega$. For $y(x)$ to form part of a solution of period ω it is necessary that $y(\omega) = y(0)$, and since the coefficients of (3) have period ω this condition is also sufficient. But the differential equation (3) is equivalent to the integral equation

$$\tfrac{1}{2}y^2(x) - \tfrac{1}{2}y^2(0) + G(x) = \alpha \int_0^x f[\xi, y(\xi)]\, d\xi,$$

and therefore, since $G(\omega) = 0$, $y(\omega) = y(0)$ if and only if

$$\int_0^\omega f[x, y(x)]\, dx = 0. \tag{5}$$

Thus (5) is a necessary and sufficient condition for $y(x)$ to have period ω.

We now ask for what values of C does there exist a solution $y = y(x, \alpha)$ of (3), defined for all sufficiently small α and having period ω in x, such that $y(x, \alpha)$ converges to $Y(x, C)$ uniformly in $0 \leqslant x \leqslant \omega$ as $\alpha \to 0$? By making $\alpha \to 0$ in the relation (5) we see that C must be a root of the equation

$$\phi(C) \equiv \int_0^\omega f[x, Y(x, C)]\, dx = 0. \tag{6}$$

Conversely, suppose $C = C^*$ is a root of this equation and write

$$Y^*(x) = Y(x, C^*).$$

For all sufficiently small α and for all y_0 sufficiently near $Y^*(0)$ the solution $y = y(x, \alpha, y_0)$ of (3) which takes the value y_0 at $x = 0$ will be defined and positive for $0 \leqslant x \leqslant \omega$. We will obtain periodic solutions converging to $Y^*(x)$ if we can choose $y_0 = y_0(\alpha)$ so that the integral

$$\psi(\alpha, y_0) \equiv \int_0^\omega f[x, y(x, \alpha, y_0)]\, dx$$

vanishes and so that $y_0(\alpha) \to Y^*(0)$ as $\alpha \to 0$. By the implicit function theorem this will be possible in one and only one way if

$$\psi_{y_0}(0, Y^*(0)) \neq 0.$$

Since

$$\psi(0, y_0) = \phi(\tfrac{1}{2}y_0^2)$$

and $\partial Y/\partial y_0 = y_0/Y$ we have

$$\psi_{y_0}(0, Y^*(0)) = Y^*(0) \int_0^\omega \frac{f_y[x, Y^*(x)]}{Y^*(x)}\, dx.$$

Hence, if $\phi(C) = 0$ and if

$$\phi'(C) = \int_0^\omega \frac{f_y[x, Y(x, C)]}{Y(x, C)}\, dx \neq 0$$

then for all sufficiently small values of α there is one and only one solution of (3) of period ω in the neighbourhood of $Y(x, C)$, and this solution converges uniformly to $Y(x, C)$ as $\alpha \to 0$.

We consider next the stability of such a periodic solution. If we put $z = y^2$, (3) is replaced by the equation

$$dz/dx = F(x, z),$$

where
$$F(x, z) = -2g(x) + 2\alpha f(x, z^{\frac{1}{2}})$$

has period ω in x. It is known—see, for example (**12**)—that a periodic solution $z(x)$ of an equation of this form is asymptotically stable or unstable for $x \to +\infty$ according as the equation of first variation

$$d\zeta/dx = F_z[x, z(x)]\zeta$$

has negative or positive characteristic exponent, that is, according as

$$\int_0^\omega F_z[x, z(x)]\, dx \lessgtr 0.$$

Returning to the original variable y it follows that for all sufficiently small α the periodic solution found above will be asymptotically stable or unstable for $x \to +\infty$ according as $\alpha\phi'(C)$ is negative or positive.

2.2. From now on it will be supposed that f_y *is everywhere negative and that for each fixed value of x there is a value of y for which $f(x, y)$ is negative.* These are essentially the assumptions made by Amerio (**6**). The first condition implies that f is a strictly decreasing function of y. Hence, by the theorem of monotonic convergence, $\phi(C)$ is a continuous, strictly decreasing function of C for which $-\infty \leqslant \phi(\infty) < 0$.

We shall show that if $\phi(C) < 0$ then for all sufficiently small α all solutions of (3) of period ω lie in the region $y < Y(x, C)$. In fact we can choose $C' < C$ so that also $\phi(C') < 0$. Let $y(x) = y(x, \alpha)$ be the solution of (3) which takes the value $(C + C')^{\frac{1}{2}}$ at $x = 0$. For all sufficiently small α this solution will be defined for $0 \leqslant x \leqslant \omega$ and will satisfy the inequalities

$$Y(x, C') < y(x) < Y(x, C).$$

Therefore, since f is a decreasing function of y,

$$\int_0^\omega f[x, y(x)]\, dx < \phi(C') < 0.$$

If $\bar{y}(x)$ is a solution of (3) of period ω, it must always be greater or always less than $y(x)$, since the continuity of f_y ensures that only one solution passes through any point in the half-plane $y > 0$. But it cannot always be

greater than $y(x)$, since this would imply

$$\int_0^\omega f[x, \bar{y}(x)]\,dx < \int_0^\omega f[x, y(x)]\,dx < 0,$$

which contradicts (5). Therefore

$$\bar{y}(x) < y(x) < Y(x, C).$$

Similarly it may be shown that if $\phi(C) > 0$ then for all sufficiently small α all solutions of (3) of period ω lie in the region $y > Y(x, C)$.

Let
$$M = \max G(x) \quad (0 \leqslant x \leqslant \omega). \tag{7}$$

If $\phi(M) > 0$ then the equation $\phi(C) = 0$ has a unique root $C = C^*$ greater than M, and $\phi'(C^*) < 0$. Since for δ small and positive

$$\phi(C^* - \delta) > 0 > \phi(C^* + \delta)$$

any solution of period ω must lie between $Y(x, C^* - \delta)$ and $Y(x, C^* + \delta)$. Moreover, by the results already established, there is exactly one solution of period ω in this region for all sufficiently small α.

On the other hand, if $\phi(M) < 0$ the equation $\phi(C) = 0$ has no root greater than M. In this case (3) has no positive solution of period ω, for all sufficiently small α. In fact any positive solution of period ω must lie in the region $y < Y(x, C)$, for every $C > M$, and hence in the region $y \leqslant Y(x, M)$. But since $Y(x, M)$ certainly vanishes at some point this is impossible.

The results which have been established can be summed up as follows:

If $\phi(M) > 0$ then for all sufficiently small α, (3) *has a unique positive solution of period ω. It is asymptotically stable or unstable for $x \to +\infty$ according as $\alpha \gtrless 0$. As $\alpha \to 0$ it converges uniformly to $Y(x, C^*)$, where C^* is the unique root greater than M of the equation $\phi(C) = 0$.*

If $\phi(M) < 0$ then for all sufficiently small α, (3) *has no positive solution of period ω.*

Let us now apply these results to the equation of a synchronous motor. The differential equation (2) will be written in the form

$$y\,dy/dx + \sin x = \alpha(\eta - y), \tag{2$'$}$$

where α and $\eta = \beta/\alpha$ are positive constants. Thus in the present case

$$g(x) = \sin x, \qquad f(x, y) = \eta - y,$$

and the various assumptions which we have made are all satisfied. Evidently

$$G(x) = 1 - \cos x = 2\sin^2 \tfrac{1}{2}x,$$

and hence

$$\phi(C) = 2\pi\eta - (2C)^{\frac{1}{2}} \int_0^{2\pi} [1 - (2/C)\sin^2 \tfrac{1}{2}x]^{\frac{1}{2}}\,dx$$
$$= 2\pi\eta - 8k^{-1}E(k),$$

where $k = (2/C)^{\frac{1}{2}}$ and $E(k)$ is the complete elliptic integral of the second kind. Moreover

$$M = \max G(x) = 2,$$

and

$$\phi(M) = 2\pi\eta - 8.$$

Substituting these values in the theorem above we arrive at once at this conclusion:

If $\eta < 4/\pi$ then for all sufficiently small α the equation (2)′ has no positive solution of period 2π, but if $\eta > 4/\pi$ then for all sufficiently small α it has one and only one such solution. This solution is asymptotically stable for $x \to +\infty$ and converges uniformly as $\alpha \to 0$ to

$$(2/k_0)(1-k_0^2 \sin^2 \tfrac{1}{2}x)^{\frac{1}{2}},$$

where k_0 is the unique root between 0 and 1 of the equation $E(k) = \frac{1}{4}\pi\eta k$.

The more complicated equation obeyed by a *salient pole* synchronous motor can be dealt with in exactly the same way.

The solution $x = x(t)$ of the original equation (1) corresponding to any solution $y = y(x)$ of the phase plane image (2) is obtained by inverting the relationship

$$t - t_0 = \int_{x_0}^{x} \frac{d\xi}{y(\xi)}.$$

In particular, to a positive periodic solution of (2) corresponds a solution of (1) which is the sum of a periodic function and a linear function. There is no difficulty in obtaining an approximate representation of this solution, using the limiting form of the periodic solution $y(x)$ as $\alpha \to 0$. However, this will not be carried out since the result will later be obtained by a different method.

2.3. To obtain a closer approximation to the periodic solution of (3) we follow the procedure of Linstedt (**9**). Suppose, for simplicity, that $f(x, y)$ is an analytic function of y. Then the periodic solution $y = y(x, \alpha)$ is analytic in α and we can write

$$y = y_0 + \alpha y_1 + \alpha^2 y_2 + \dots,$$

where the coefficients y_i all have period ω. It follows that

$$y' = y_0' + \alpha y_1' + \alpha^2 y_2' + \dots,$$

$$\alpha f(x, y) = \alpha f(x, y_0) + \alpha^2 f_y(x, y_0) y_1 + \dots.$$

Substituting these expansions in (3) and equating coefficients of like powers of α we get

$$\begin{aligned} y_0 y_0' + g(x) &= 0, \\ y_0 y_1' + y_1 y_0' &= f(x, y_0), \\ y_0 y_2' + y_1 y_1' + y_2 y_0' &= f_y(x, y_0) y_1, \\ \cdots & \cdots \end{aligned}$$

The first equation gives us

$$y_0 = 2^{\frac{1}{2}}[c_0 - G(x)]^{\frac{1}{2}}$$

for some constant c_0. From the second equation we get

$$y_0 y_1 = c_1 + \int_0^x f[\xi, y_0(\xi)]\, d\xi$$

for some constant c_1. Since y_1 must have period ω we arrive, as before, at the equation

$$\int_0^\omega f[x, y_0(x)]\, dx = 0$$

to determine c_0. The third equation gives us

$$y_0 y_2 + \tfrac{1}{2} y_1^2 = c_2 + \int_0^x f_y[\xi, y_0(\xi)] y_1(\xi)\, d\xi$$

for some constant c_2. Since y_2 must have period ω we obtain the equation

$$\int_0^\omega f_y[x, y_0(x)] y_1(x)\, dx = 0$$

to determine c_1. And so on.

For the equation of a synchronous motor $\omega = 2\pi$ and $f_y \equiv -1$. Hence the last equation can be written

$$\int_{-\pi}^{\pi} y_1(x)\, dx = 0. \tag{*}$$

The equations for y_0 and y_1 become

$$y_0 = (2/k_0)(1 - k_0^2 \sin^2 \tfrac{1}{2}x)^{\frac{1}{2}},$$

$$y_0 y_1 = c_1 + \eta x - (2/k_0) \int_0^x (1 - k_0^2 \sin^2 \tfrac{1}{2}\xi)^{\frac{1}{2}}\, d\xi.$$

Since y_0 is an even function the condition (*) reduces to $c_1 = 0$. Thus, neglecting powers of α above the first, the periodic solution of (2)′ is given by

$$y = (2/k_0)\Delta(x) + \alpha\left\{\tfrac{1}{2}\eta k_0 x - \int_0^x \Delta(\xi)\, d\xi\right\}\Big/\Delta(x),$$

where k_0 has its former meaning and $\Delta(x) = (1 - k_0^2 \sin^2 \tfrac{1}{2}x)^{\frac{1}{2}}$.

3. The method of averaging

3.1. The method described in the previous section has the disadvantage that it provides information only about the periodic solutions. To deduce the behaviour of all the solutions the qualitative results of Tricomi and Amerio must be used. We are going to apply now a different method which is not subject to this restriction, but which has no pretence to

rigour. It is a generalization of the method which Kryloff and Bogoliuboff (**11**)—following van der Pol—have applied to the quasi-linear equation

$$\ddot{x}+x = \epsilon f(x,\dot{x}). \tag{8}$$

For purposes of comparison we will first briefly describe their method. The equation

$$\ddot{x}+x = 0,$$

obtained from (8) by putting $\epsilon = 0$, has the solutions $x = a\cos(t+\theta_0)$, where a and θ_0 are arbitrary constants. Kryloff and Bogoliuboff therefore introduce new *variables* a and ψ defined by the equations

$$x = a\cos\psi,$$

$$\dot{x} = -a\sin\psi.$$

It is then easily shown that the second-order equation (8) is replaced by the pair of first-order equations

$$\left.\begin{aligned} \dot{a} &= -\epsilon\sin\psi f(a\cos\psi,\ -a\sin\psi) = \epsilon F(a,\psi) \\ \dot{\psi} &= 1-\epsilon a^{-1}\cos\psi f(a\cos\psi,\ -a\sin\psi) = 1+\epsilon G(a,\psi) \end{aligned}\right\}. \tag{9}$$

The functions $F(a,\psi)$ and $G(a,\psi)$ are periodic in ψ with period 2π. They can therefore be broken up (uniquely) into two parts

$$F(a,\psi) = F_1(a)+\frac{\partial}{\partial\psi}F_2(a,\psi),$$

$$G(a,\psi) = G_1(a)+\frac{\partial}{\partial\psi}G_2(a,\psi),$$

where

$$F_1(a) = \frac{1}{2\pi}\int_0^{2\pi} F(a,\psi)\,d\psi,$$

$$G_1(a) = \frac{1}{2\pi}\int_0^{2\pi} G(a,\psi)\,d\psi$$

are the mean values of F and G respectively, and where F_2, G_2 are periodic in ψ with period 2π and have *zero* mean value. Suppose now that ϵ is small. Then the equations (9) show that formally

$$\dot{a} = O(\epsilon), \qquad \dot{\psi} = 1+O(\epsilon).$$

Therefore

$$d/dt = \frac{\partial}{\partial\psi}+O(\epsilon).$$

Thus the equations (9) can be rewritten in the form

$$\left.\begin{aligned} \dot{a} &= \epsilon F_1(a)+\epsilon\frac{d}{dt}F_2(a,\psi)+O(\epsilon^2) \\ \dot{\psi} &= 1+\epsilon G_1(a)+\epsilon\frac{d}{dt}G_2(a,\psi)+O(\epsilon^2) \end{aligned}\right\}. \tag{9'}$$

The approximation procedure consists first of all in neglecting the terms of order ϵ^2 in these equations. Secondly, the *average trends* $\bar{a}$ and $\bar{\psi}$ of a and ψ are obtained by neglecting the small periodic terms involving F_2 and G_2. Thus $\bar{a}$ and $\bar{\psi}$ are defined as the solution of the system of equations

$$\dot{\bar{a}} = \epsilon F_1(\bar{a}),$$

$$\dot{\bar{\psi}} = 1+\epsilon G_1(\bar{a}).$$

These values are then substituted for a and ψ in the right-hand side of (9)′. The resulting equations can be integrated at once and give

$$a = \bar{a}+\epsilon F_2(\bar{a},\bar{\psi}),$$

$$\psi = \bar{\psi}+\epsilon G_2(\bar{a},\bar{\psi}).$$

These are the complete first-order approximations. However, for many purposes the motion is already adequately described by the average trends $\bar{a}$ and $\bar{\psi}$. The effect of the second terms is simply to add small oscillations about this mean motion.

3.2. We return now to the equation of a synchronous motor and write it in a form analogous to (8), namely,

$$\ddot{x}+\sin x = \alpha f(\dot{x}), \tag{10}$$

where

$$f(\dot{x}) = \eta-\dot{x}. \tag{11}$$

To put the equation in a more suitable form for approximating we apply the method of variation of constants to the well-known solution, in terms of elliptic functions, of the equation for an ordinary pendulum:

$$\ddot{x}+\sin x = 0.$$

Let

$$W = W(t) = \sin^2\tfrac{1}{2}x+\tfrac{1}{4}\dot{x}^2; \tag{12}$$

W is constant for the pendulum equation, $2W$ being its total energy apart from an arbitrary additive constant. We take different forms for the solution of (10) according as $W \gtrless 1$, corresponding to the two types of pendulum motion—oscillation and revolution.

If $W > 1$ and $\dot{x} > 0$ we write the solution in the form

$$\sin\tfrac{1}{2}x = \operatorname{sn} u, \quad \cos\tfrac{1}{2}x = \operatorname{cn} u, \quad \tfrac{1}{2}\dot{x} = k^{-1}\operatorname{dn} u, \tag{13}$$

where k and u are functions of the independent variable t, and k is the modulus of the elliptic functions. For this change of variables to be a legitimate one the transformation must be invertible. But from the definition of W we have

$$W = \operatorname{sn}^2 u+k^{-2}\operatorname{dn}^2 u = k^{-2}, \tag{14}$$

and consequently the inverse transformation is given explicitly by the equations

$$k = (\sin^2 \tfrac{1}{2}x + \tfrac{1}{4}\dot{x}^2)^{-\frac{1}{2}},$$

$$u = \int_0^{\frac{1}{2}x} (1-k^2 \sin^2\xi)^{-\frac{1}{2}}\, d\xi.$$

From (14) we get by differentiation

$$\frac{d}{dt}(k^{-2}) = \tfrac{1}{2}\dot{x}(2\sin \tfrac{1}{2}x \cos \tfrac{1}{2}x + \ddot{x}).$$

By (10) the right-hand side is equal to $\frac{1}{2}\alpha\dot{x}f(\dot{x})$. Hence

$$\frac{d}{dt}(k^{-2}) = \alpha k^{-1}\operatorname{dn} u\, f(2k^{-1}\operatorname{dn} u). \tag{15}$$

Also, differentiating the first equation (13), we get

$$\frac{d}{dt}(\operatorname{sn} u) = \tfrac{1}{2}\dot{x}\cos \tfrac{1}{2}x = k^{-1}\operatorname{cn} u \operatorname{dn} u.$$

But by the formulae for the total differentiation of elliptic functions (13)

$$\frac{d}{dt}(\operatorname{sn} u) = \operatorname{cn} u \operatorname{dn} u\left\{\frac{du}{dt} - \tfrac{1}{2}\operatorname{Sd} u\,.\,\frac{d}{dt}(k^2)\right\},$$

where, in Neville's notation,

$$\operatorname{Sd} u = \int_0^u \frac{\operatorname{sn}^2 u}{\operatorname{dn}^2 u}\, du.$$

Combining this with the previous equation we get

$$\frac{du}{dt} = k^{-1} + \tfrac{1}{2}\operatorname{Sd} u\,.\,\frac{d}{dt}(k^2). \tag{16}$$

Thus the second-order equation (10) in the variable x is equivalent to the two first-order equations (15) and (16) in the variables k and u.

The functions

$$F(k,u) = k^{-1}\operatorname{dn} u\, f(2k^{-1}\operatorname{dn} u),$$

$$G(k,u) = \operatorname{Sd} u - (u/K)\operatorname{Sd} K$$

are periodic in u with period $4K$, where $K = K(k)$ is the complete elliptic integral of the first kind. The same also holds for their product. Moreover, since FG is an odd function of u its mean value is zero.

Suppose now that α is small. From the equations (15) and (16) we have

$$\frac{d}{dt}(k^{-2}) = O(\alpha), \qquad \frac{du}{dt} = k^{-1} + O(\alpha).$$

Proceeding as in the method of slowly varying amplitude and phase, and

using the fact that $\operatorname{Sd} K = 2dK/d(k^2)$ (**13**), it follows that

$$\frac{d}{dt}(k^{-2}) = \alpha F_1(k) + \alpha \frac{d}{dt} F_2(k,u) + O(\alpha^2),$$

$$\frac{du}{dt} = k^{-1} + \frac{u}{K}\frac{dK}{dt} + \alpha \frac{d}{dt} G_2(k,u) + O(\alpha^2).$$

Here
$$F_1(k) = \frac{1}{4K} \int\limits_{-2K}^{2K} k^{-1} \operatorname{dn} u\, f(2k^{-1} \operatorname{dn} u)\, du$$

is the mean value of F, while F_2 and G_2 are periodic in u with period $4K$ and mean value zero. To obtain the average trends $\bar{k}$ and $\bar{u}$ of k and u we neglect the terms of order ϵ^2 and the small periodic terms involving F_2 and G_2. Thus $\bar{k}$ and $\bar{u}$ satisfy the equations

$$\frac{d}{dt}(\bar{k}^{-2}) = \alpha F_1(\bar{k}), \tag{17}$$

$$\frac{d\bar{u}}{dt} = \bar{k}^{-1} + \frac{\bar{u}}{\bar{K}}\frac{d\bar{K}}{dt}. \tag{18}$$

Having obtained $\bar{k}$ and $\bar{u}$ the complete first-order approximations for k and u can immediately be found. However, since the complete expressions are complicated and add nothing essential, in what follows we shall restrict the discussion to the average trends $\bar{k}$ and $\bar{u}$. Moreover, for convenience of writing we will drop the bars and write simply k and u.

The solution of the first-order linear differential equation (18) is

$$u = K(t) \int^{t} \frac{d\tau}{k(\tau)K(\tau)}. \tag{19}$$

Consider next equation (17). For the case of a synchronous motor, that is for $f(\dot{x})$ given by (11),

$$F_1(k) = \frac{1}{4K} \int\limits_{-2K}^{2K} k^{-1} \operatorname{dn} u(\eta - 2k^{-1} \operatorname{dn} u)\, du,$$

or, since $\operatorname{dn} u = \operatorname{dn}(-u) = \operatorname{dn}(2K-u)$,

$$F_1(k) = \frac{1}{K} \int\limits_{0}^{K} k^{-1} \operatorname{dn} u(\eta - 2k^{-1} \operatorname{dn} u)\, du.$$

The integral here is easily evaluated—see, for example, (**14**)—and we find that

$$\frac{d}{dt}(k^{-2}) = 2\alpha(\tfrac{1}{4}\pi\eta k - E)/k^2 K, \tag{20}$$

where $E = E(k)$ is the complete elliptic integral of the second kind. This

can be written in the form

$$\frac{\frac{1}{2}k^{-3}K\,d(k^2)}{\frac{1}{4}\pi\eta-k^{-1}E} = -\alpha\,dt.$$

But using the formulae for the derivatives of complete elliptic integrals we have

$$d(k^{-1}E)/d(k^2) = -\tfrac{1}{2}k^{-3}K.$$

Hence the solution of (20) is given by

$$k^{-1}E = \tfrac{1}{4}\pi\eta+c_1e^{-\alpha t}, \tag{21}$$

where c_1 is a constant.

Now E decreases when k increases and takes the value 1 at $k = 1$. Hence if $\eta < 4/\pi$, the function $E-\frac{1}{4}\pi\eta k$ is always positive. In this case it follows from (20) that $W = k^{-2}$ decreases as t increases, and from (21) that after a finite time we must have $W = 1$.

On the other hand if $\eta > 4/\pi$ the equation $E = \frac{1}{4}\pi\eta k$ has a unique root k_0 between 0 and 1, and $d(k^{-2})/dt \gtrless 0$ according as $k \gtrless k_0$. Hence k converges monotonically to k_0, whether initially it is greater or less than this value. By (21) the time taken to reach the limit k_0 is infinite. Moreover, writing (21) in the form

$$\tfrac{1}{4}\pi\eta-\tfrac{1}{2}k_0^{-3}K_0(k^2-k_0^2)+O(k-k_0)^2 = \tfrac{1}{4}\pi\eta+c_1e^{-\alpha t},$$

we see that for $t \to +\infty$

$$k-k_0 \doteqdot -\frac{k_0^2}{K_0}c_1e^{-\alpha t}.$$

Similarly for $t \to +\infty$

$$u = K(t)\int^{t}\frac{d\tau}{k(\tau)K(\tau)} = \alpha^{-1}K(t)\int^{k}\frac{k^{-2}\,dk}{E-\frac{1}{4}\pi\eta k}$$

$$\doteqdot (t-t_0)/k_0+(At+B)e^{-\alpha t},$$

where t_0 is a constant of integration and A and B are constants whose values can easily be determined.

3.3. These solutions are for the case $W > 1$, $\dot{x} > 0$. If $W > 1$ and $\dot{x} < 0$ the form of the solution must be modified. We then take

$$\sin\tfrac{1}{2}x = -\operatorname{sn}u, \quad \cos\tfrac{1}{2}x = \operatorname{cn}u, \quad \tfrac{1}{2}\dot{x} = -k^{-1}\operatorname{dn}u.$$

Equations (14) and (16) remain unchanged, but (15) is replaced by

$$\frac{d}{dt}(k^{-2}) = -\alpha k^{-1}\operatorname{dn}u\,f(-2k^{-1}\operatorname{dn}u).$$

Consequently the equation (20) for $\bar{k}$ becomes

$$\frac{d}{dt}(k^{-2}) = -2\alpha(\tfrac{1}{4}\pi\eta k+E)/k^2K.$$

The solution of this equation is obtained by changing the sign of η in (21),

$$k^{-1}E = -\tfrac{1}{4}\pi\eta+c_1e^{-\alpha t}.$$

It follows that $W = k^{-2}$ decreases as t increases and that after a finite time we have $W = 1$.

3.4. Suppose finally that $W < 1$. In this case we write the solution in the form

$$\sin\tfrac{1}{2}x = k\,\mathrm{sn}\,u, \quad \cos\tfrac{1}{2}x = \mathrm{dn}\,u, \quad \tfrac{1}{2}\dot{x} = k\,\mathrm{cn}\,u, \tag{22}$$

where again k and u are functions of t, and k is the modulus of the elliptic functions. For definiteness it will be supposed that initially $-\pi < x < \pi$. From the definition of W we have

$$W = k^2\,\mathrm{sn}^2 u + k^2\,\mathrm{cn}^2 u = k^2. \tag{23}$$

Proceeding as for the case $W > 1$ we find that

$$\frac{d}{dt}(k^2) = \alpha k\,\mathrm{cn}\,u\, f(2k\,\mathrm{cn}\,u),$$

$$\frac{du}{dt} = 1 + \tfrac{1}{2}\left\{\mathrm{Sd}\,u - \frac{\mathrm{sn}\,u}{k^2\,\mathrm{cn}\,u\,\mathrm{dn}\,u}\right\}\cdot\frac{d}{dt}(k^2).$$

By splitting up the right-hand sides in the same way as before we get

$$\frac{d}{dt}(k^2) = \alpha F_1(k) + \alpha\frac{d}{dt}F_2(k,u) + O(\alpha^2),$$

$$\frac{du}{dt} = 1 + \frac{u}{K}\frac{dK}{dt} + \alpha\frac{d}{dt}G_2(k,u) + O(\alpha^2),$$

where
$$F_1(k) = \frac{1}{4K}\int_{-2K}^{2K} k\,\mathrm{cn}\,u\, f(2k\,\mathrm{cn}\,u)\,du$$

and F_2, G_2 are periodic in u with period $4K$ and mean value zero. For the equation of a synchronous motor, that is for $f(\dot{x})$ given by (11),

$$F_1(k) = \frac{1}{4K}\int_{-2K}^{2K} k\,\mathrm{cn}\,u(\eta - 2k\,\mathrm{cn}\,u)\,du = -\frac{2}{K}\int_0^K k^2\,\mathrm{cn}^2 u\,du$$
$$= -2(E - k'^2 K)/K,$$

where $k'^2 = 1 - k^2$. Thus the average trends $\bar{k}$ and $\bar{u}$ satisfy the equations

$$\frac{d}{dt}(k^2) = -2\alpha(E - k'^2K)/K, \tag{24}$$

$$\frac{du}{dt} = 1 + \frac{u}{K}\frac{dK}{dt}. \tag{25}$$

The solution of the first-order linear differential equation (25) is

$$u = K(t)\int^t \frac{d\tau}{K(\tau)}. \tag{26}$$

Also, from the formulae for the derivatives of complete elliptic integrals,

$$d(E-k'^2K)/d(k^2) = \tfrac{1}{2}K.$$

Hence the solution of the separated variables equation (24) is given by

$$E-k'^2K = c_2 e^{-\alpha t}, \tag{27}$$

where c_2 is a constant.

Now the function $E-k'^2K$ increases from 0 to 1 as k increases from 0 to 1. Therefore, by (24), $W = k^2$ decreases as t increases, and by (27) $k^2 \to 0$ as $t \to +\infty$. Since $|\sin \frac{1}{2}x| \leqslant k < 1$ it follows that x remains inside the interval $(-\pi, \pi)$ and tends to zero as $t \to +\infty$. (In general x will converge to the nearest even multiple of π.) When t is large we have from (27)

$$k^2 \doteqdot \frac{4c_2}{\pi} e^{-\alpha t}.$$

Similarly from the equation

$$u = K(t) \int^{t} \frac{d\tau}{K(\tau)} = -\tfrac{1}{2}\alpha^{-1}K(t) \int \frac{d(k^2)}{E-k'^2K}$$

we get

$$u \doteqdot t-t_0+\frac{c_2}{\pi}(t-t_0+\alpha^{-1})e^{-\alpha t},$$

where t_0 is a constant of integration.

It only remains to piece together the various solutions which have been obtained. Suppose that a solution for which $W > 1$ initially arrives at $W = 1$ when $t = t_1$. There are three possibilities consistent with the differential equations (20) and (24) which W satisfies in the regions $W \gtrless 1$:

(i) W passes immediately into the domain $W < 1$,

(ii) W remains equal to 1 for all $t \geqslant t_1$,

(iii) W remains equal to 1 for an interval $t_1 \leqslant t \leqslant t_2$ and then passes into the domain $W < 1$.

The approximate equations naturally do not enable one to decide between these possibilities. However, by appealing to the exact equation (1) we can show that W cannot be constantly equal to 1 throughout an interval of time, and on this basis we can exclude cases (ii) and (iii). Hence our conclusions may be summarized thus:

If $\eta < 4/\pi$ all solutions converge to a finite limit as $t \to +\infty$, but if $\eta > 4/\pi$ only those solutions converge for which initially $\dot{x} < 0$ or $\sin^2 \frac{1}{2}x + \frac{1}{4}\dot{x}^2 \leqslant 1$. The remaining solutions behave asymptotically as though x were the angle made with the downward vertical by a simple pendulum of length g which performs complete revolutions in a time $2k_0K(k_0)$, where k_0 is the unique root between 0 and 1 of the equation $E(k) = \frac{1}{4}\pi\eta k$.†

† Since k_0 decreases as η increases the period of revolution is a monotonic decreasing function of η.

These results agree with the qualitative results stated in the introduction. There are two types of solution, the convergent solutions which tend to a finite limit as $t \to +\infty$, and the divergent solutions which behave asymptotically as the sum of a linear function and a periodic function. Moreover there is a critical value η_c of the ratio $\eta = \beta/\alpha$ such that all solutions are convergent if and only if $\eta < \eta_c$. The limit of the convergent solutions is given as zero (mod 2π), whereas the exact value is easily shown to be $\sin^{-1}\beta$ (**4**). Thus the approximations give the limit of the exact value for $\beta \to 0$. The initial conditions which characterize the divergent solutions when $\eta > \eta_c$ may also be regarded as the limit for $\beta \to 0$ of the known qualitative results. Finally, as proved in section 2, the value $4/\pi$ for the critical value η_c and the value $2k_0K_0$ for the time period associated with the divergent solutions are the limits for $\beta \to 0$ ($\beta/\alpha =$ constant) of the exact values of these quantities.

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A FURTHER NOTE ON THE CALCULATION OF HEAT TRANSFER THROUGH THE AXISYMMETRICAL LAMINAR BOUNDARY LAYER ON A CIRCULAR CYLINDER

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SUMMARY

By using a Kármán–Pohlhausen method the distribution of local rate of heat transfer is evaluated for the case of air flow in an axisymmetrical laminar boundary layer on a heated circular cylinder, the temperature of the cylinder being independent of downstream distance. This calculation serves to link the numerical values obtained by Seban, Bond, and Kelly for small downstream distances to those obtained by Bourne and Davies for large downstream distances.

1. In a previous paper by Bourne and Davies (**1**) the influence of curvature on the calculated values of local rate of heat transfer from a heated surface was demonstrated by considering the problem of forced convection from a heated circular cylinder into the surrounding, incompressible, axisymmetric laminar boundary layer produced by a uniform stream. An exact solution of the temperature equation had already been given by Seban and Bond (**2**), with a correction by Kelly (**3**), in the form of a series, valid for small values of $\nu x/Ua^2$ ($\leqslant$ about 0·04), where U denotes uniform main-stream velocity, x downstream distance, ν kinematic viscosity, and a radius of the cylinder. Bourne and Davies obtained an asymptotic series solution, which is valid at large values of $\nu x/Ua^2$ ($\geqslant$ about 100) and is based on the type of analysis given by Glauert and Lighthill (**4**).

In order to bridge the gap, for the intermediate values of $\nu x/Ua^2$, between the results given by these two series solutions, Bourne and Davies applied in a modified form an approximate method of calculation which they had previously introduced for the case of heat transfer from a flat plate. This method was based on a power law approximation to the Pohlhausen velocity profile and, although capable of application to the general case of arbitrary distribution of cylinder temperature, suffers from the disadvantage of requiring a separate power law fit for each point on the bridge. The maximum deviation of the power law from the appropriate Pohlhausen profile (over 95 per cent of the boundary layer thickness)

increases from about 4 per cent at the smaller intermediate values of $\nu x/Ua^2$ to about 10 per cent at the larger values. In the latter region the power law method is unlikely to lead to really accurate results.†

A check on the results obtained by this method is now achieved in this paper by an alternative calculation. A Kármán–Pohlhausen approximation is employed, following the method described by Glauert and Lighthill in calculating the distribution of skin-friction on the cylinder, and the distribution of heat transfer obtained by numerical integration of a first-order linear differential equation. This method is likely to be more accurate at the larger intermediate values of $\nu x/Ua^2$.

2. The temperature integral relation, expressing conservation of heat between two planes perpendicular to the axis of the cylinder, is

$$\rho c_p \frac{d}{dx}\int_0^{\delta_H} u(T-T_0)(a+y)\,dy = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} a, \tag{1}$$

where T is the temperature of the air stream at normal distance y from the surface of the cylinder, T_0 the temperature of the main stream, u the downstream velocity, δ_H the thickness of the temperature boundary layer at downstream distance x, ρ the fluid density, c_p the specific heat of the fluid at constant pressure, k the thermal conductivity of the fluid.

The discussion given by Glauert and Lighthill (**4**) shows that their Pohlhausen velocity form represents flow conditions near the cylinder surface fairly accurately. This may be written in the form

$$\left.\begin{aligned} \frac{u}{U} &= \frac{1}{\alpha}\ln(1+y/a) \quad \text{for} \quad y \leqslant \delta = a(e^{\alpha}-1) \\ \text{and} \qquad u/U &= 1 \quad \text{for} \quad y \geqslant \delta \end{aligned}\right\}, \tag{2}$$

where U is the mainstream velocity, δ is the thickness of the momentum boundary layer at downstream distance x, and α is a function of $\nu x/Ua^2$ given numerically by Glauert and Lighthill. Their argument is immediately applicable in the context of heat flow to show that the temperature distribution is well represented by the forms

$$\left.\begin{aligned} \frac{T_1-T}{T_1-T_0} &= \frac{1}{\beta}\ln(1+y/a) \quad \text{for} \quad y \leqslant \delta_H = a(e^{\beta}-1) \\ \text{and} \qquad \frac{T_1-T}{T_1-T_0} &= 1 \quad \text{for} \quad y \geqslant \delta_H \end{aligned}\right\}, \tag{3}$$

where T_1 is the temperature of the cylinder surface and β is a function of $\nu x/Ua^2$ to be determined, corresponding to α in equation (2). Substituting

† An accuracy of about 5 per cent was suggested by Bourne and Davies (**1**).

(2) and (3) into (1) leads to the relation

$$\frac{d}{dx}\left\{\frac{1}{\alpha}\int\limits_0^{\delta}\left[1-\frac{1}{\beta}\ln(1+y/a)\right][\ln(1+y/a)](a+y)\,dy+\right.$$

$$\left.+\int\limits_{\delta}^{\delta_H}\left[1-\frac{1}{\beta}\ln(1+y/a)\right](a+y)\,dy\right\}=\frac{\kappa}{U\beta} \qquad (4)$$

for $\delta_H \geqslant \delta$ (corresponding to values of σ, the Prandtl number, $\leqslant 1$), where κ is the thermal diffusivity of the fluid. A similar relation can be obtained for $\delta_H < \delta$ (corresponding to $\sigma > 1$). Writing $y = a(e^{s}-1)$ in equation (4) and integrating yields

$$\frac{d}{dx}\left[\frac{1}{\alpha\beta}\{e^{2\alpha}[\tfrac{1}{2}\beta(\alpha-\tfrac{1}{2})-\tfrac{1}{2}(\alpha^2-\alpha+\tfrac{1}{2})]+\tfrac{1}{4}(1+\beta)\}+\frac{1}{\beta}\{\tfrac{1}{4}e^{2\beta}-\tfrac{1}{2}e^{2\alpha}(\beta-\alpha+\tfrac{1}{2})\}\right]$$

$$=\frac{\kappa}{U\beta a^2}. \qquad (5)$$

Differentiating and using the relation

$$\alpha^{-2}\{(2\alpha^2-3\alpha+2)e^{2\alpha}-(\alpha+2)\}\frac{d\alpha}{dx}=\frac{4\nu}{Ua^2}, \qquad (6)$$

given by Glauert and Lighthill, we find finally that

$$\frac{d\beta}{d\alpha}[\beta^{-1}(e^{2\alpha}-e^{2\beta})+2e^{2\beta}-\alpha\beta^{-1}\{\alpha^{-2}-e^{2\alpha}(\alpha^{-2}-2\alpha^{-1})\}]$$

$$=2(2\sigma^{-1}-1)e^{2\alpha}+(1+\beta-2\sigma^{-1}-\alpha\sigma^{-1})\{\alpha^{-2}-e^{2\alpha}(\alpha^{-2}-2\alpha^{-1})\}. \qquad (7)$$

This equation is now in standard form and is to be solved subject to the end condition $\beta = 0$ at $\alpha = 0$, and at *large* downstream distances the values of β must run smoothly into the values given by the asymptotic solution. This enables β to be calculated as a function of α, which is already given numerically as a function of $\nu x/Ua^2$ by Glauert and Lighthill. The local rate of heat transfer Q, per unit length of the cylinder, is then given by $2\pi k(T_1-T_0)/\beta$.

3. For small values of $\nu x/Ua^2$, we note that α is given in terms of a series in the variable $\nu x/Ua^2$ by Glauert and Lighthill (**4**), and β in terms of another series in $(\nu x/Ua^2)$ by Seban and Bond (**3**). In this region the values of α and β can thus be calculated for selected values of $\nu x/Ua^2$ and the values of β corresponding to, say, $\alpha = 0{\cdot}05$, $0{\cdot}10$, and $0{\cdot}15$ can be obtained by interpolation. With these initial values and the condition $\alpha = 0, \beta = 0$, equation (7) can be integrated numerically using a standard step-by-step forward integration procedure. In this paper the Milne–Simpson method has been used, the steps in α being $0{\cdot}05$ initially and $0{\cdot}1$ for values of α greater than $0{\cdot}3$, and the integration taken as far as $\alpha = 3{\cdot}0$

(corresponding to $\log_{10}(\nu x/Ua^2) = 2$, in the range of the asymptotic series solution). The results shown in Table 1 were obtained by interpolation, using the relation between α and $\nu x/Ua^2$ given by Glauert and Lighthill.

TABLE 1

Calculated values of local heat transfer rates in air, per unit length of a circular cylinder, obtained by two methods

$\log_{10}\left(\frac{\nu x}{Ua^2}\right)$	$\log_{10}\left[\frac{Q}{k(T_1-T_0)}\right]$	
	Values obtained by numerical integration of the (Kármán–Pohlhausen) differential equation (7)	*Values obtained by the power law method of Bourne and Davies*
−1·0	0·97	0·96
−0·5	0·82	0·79
0	0·68	0·64
+0·5	0·56	0·52
+1·0	0·46	0·42

The numerical results for $Q/k(T_1-T_0)$, shown in Table 1 over the intermediate range of values of $\nu x/Ua^2$ (corresponding to the bridge), are found to run smoothly into the results given by the asymptotic series solution for $\log_{10}\nu x/Ua^2 > 2$, and are seen to be in fairly close agreement with the results obtained by the power law method employed previously by Bourne and Davies (**1**). As noted above, the error involved in representing the Pohlhausen profile by a power law increases, as $\nu x/Ua^2$ increases, and the power law method, when applied at the downstream end of the bridge, is therefore not likely to lead to very good accuracy. On the other hand, the results given in the second column of Table 1, by numerical integration of (7), are of the same order of accuracy as those given for skin friction by Glauert and Lighthill, and we suggest they should replace the previously calculated values given in column 3 of the table.

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NOTE ON ALLEN AND SOUTHWELL'S PAPER 'RELAXATION METHODS APPLIED TO DETERMINE THE MOTION, IN TWO DIMENSIONS, OF A VISCOUS FLUID PAST A FIXED CYLINDER'

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SUMMARY

In the present note, the results of Allen and Southwell's paper, which dealt with the motion of a viscous fluid past a fixed circular cylinder by applying relaxation techniques, have been compared with the results hitherto known, both theoretical and experimental, and it has thus been shown that there are some discrepancies between Allen and Southwell's results and the other authors'. It seems that the origin of these discrepancies may be found in the unsatisfactory treatment of the point at infinity or in the use of coarse meshes in Allen and Southwell's study.

1. RECENTLY Allen and Southwell (**1**) have dealt with the two-dimensional steady flow of a viscous fluid past a circular cylinder, by the use of the relaxation method. They have thus obtained numerical solutions for various values of Reynolds number, $R = 0, 1, 10, 100, 1000$, assuming the flow to be laminar. We can infer the characteristic features of their solutions from many figures in their paper, although the details of their computation were omitted, e.g. the treatment of the point at infinity, the mesh length, the accuracy of their solutions, etc.

There are both theoretical and experimental papers dealing with the same problem, especially for the lower Reynolds number régime where the flow is stationary. Thus, Thom ($R = 10, 20$) (**2**) and the present author ($R = 40$) (**3**) obtained the numerical solutions for the same problem by the use of Liebmann's method. As the starting form of such a numerical solution, the present author adopted the results derived from his study on the problem by means of the Galerkin method (**4**). The author also studied the limiting form of the laminar solution when the Reynolds number becomes infinitely large (**5**). For the very low Reynolds number régime, Tomotika and Aoi (**6**, **7**, **8**) treated the same problem using Oseen's approximation. On the other hand, Thom (**2**), Kovásznay (**9**), Homann (**10**), and Taneda (**11**) studied the flow experimentally.

The characteristic features of Allen and Southwell's solutions seem to be inconsistent with the results hitherto known:

(i) In Fig. 1 are shown the values of the characteristic length a/r of the standing vortices behind the cylinder (cf. Fig. 2) which have been obtained from Allen and Southwell's paper, in comparison with the other results. It will easily be seen that Allen and Southwell's solutions show a quite different feature from the others'. According to the author's

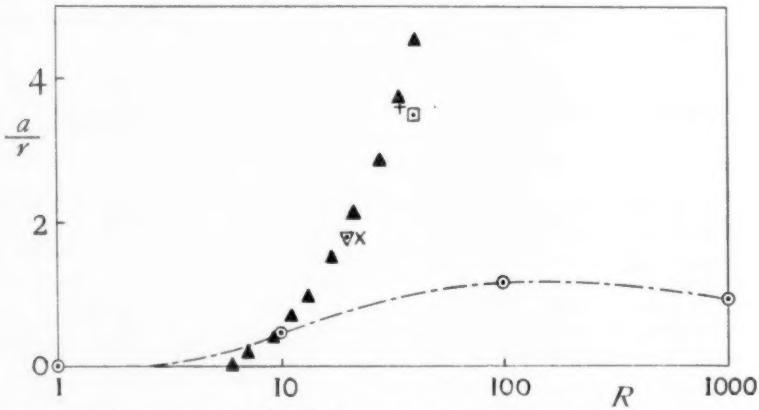

FIG. 1. The characteristic length of the standing vortices behind the circular cylinder.

THEORETICAL: ⊙ Allen & Southwell (**1**); ▽ Thom (**2**); ⊡ Kawaguti (**3**)

EXPERIMENTAL: × Thom (**2**); + Homann (**10**); ▲ Taneda (**11**)

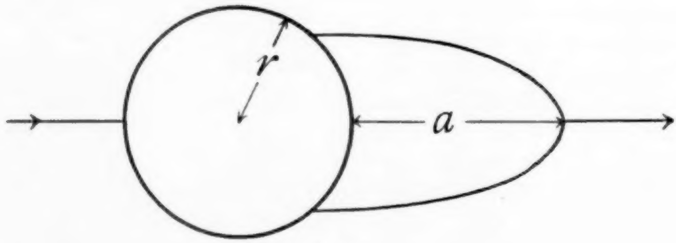

FIG. 2

opinion, the ratio a/r should become infinitely large as the Reynolds number increases indefinitely (**3, 4**).

(ii) In Fig. 7, Allen and Southwell gave the pressure distributions over the surface of the cylinder. These pressure distributions have their maxima at points other than the forward stagnation point where the pressure distributions obtained by the others show their maxima.

In view of these discrepancies, the author cannot but be suspicious of the accuracy of their results. The origin of the discrepancies between Allen and Southwell's and the other authors' results may be found in the unsatisfactory treatment of the point at infinity or in the use of coarse meshes in Allen and Southwell's study.

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L. S. D. MORLEY, 'An improvement on Donnell's approximation for thin-walled circular cylinders'. Vol. XII, Pt. 1, 1959.

References **3** and **4** refer to the *J. Appl. Mech.*

V. C. LIU, 'On the separation of gas mixtures by suction of the thermal-diffusion boundary layer'. Vol. XII, Pt. 1, 1959.

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